Exploring The Thirteen Colorful Variations of Gutherie's Four-Color Conjecture

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Abstract

Coloring is an important part of graph theory. Historically, it was thought that only four colors could be the minimal number of colors. This paper discusses The Four Color theorem and how the Four Color Theorem is applied graphs. This paper gives an overview of several different definitions involved with graphs and shows how to create a dual graph. This paper also discusses how a graph of 12 regions has at least one region bounded by less than five edges. This paper includes several figures which include graphs, dual graphs, and different colorings. This paper also provides a proof which shows mathematically why a graph of 12 regions has at least one region bounded by less than five edges.

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1 Introduction

Coloring is an important graph theory topic. When discussing coloring in a graph theory class, it starts with coloring a map. When thinking about coloring a map, it is interesting to notice that while many different colors can be used, but also there is the possibility of even fewer colors being needed. Historically, it was thought to be four colors. This idea of only needing four colors to color any map is incredibly hard to prove. This paper will be sharing some of the mathematics that are involved with coloring graphs.

2 Four Color Theorem

The Four Color Theorem is any given separation of a plane into adjacent regions, producing a map, which require no more than four colors to color the regions such that no two adjacent regions are the same color.

2.1 Example

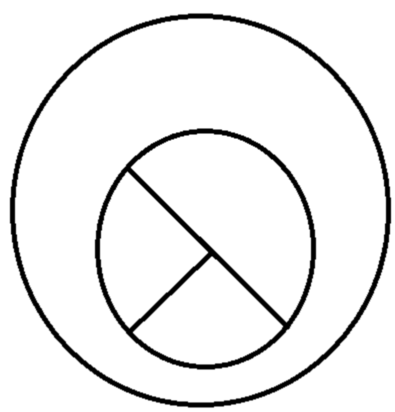

Figure 1: Map

This is an image of an uncolored map with five different regions. These regions include the surrounding area of the map.

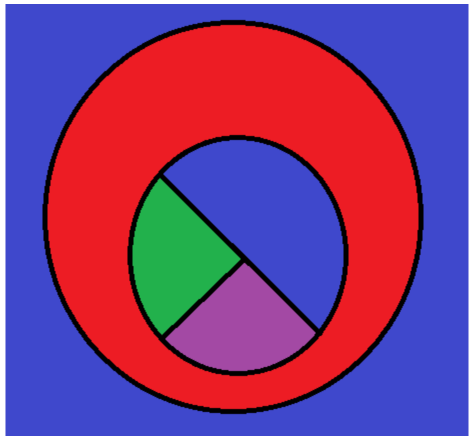

Figure 2: Colored Map

This is an image of the same map, but colored with only four colors. No two adjacent regions are the same color meaning the blue region is not touching the other blue region.

2.2 Definitions

Definition 2.1. *A graph is a pair of* (V, E) *where* V *is a finite nonempty set called the set of vertices,* E *is a finite set called the set of edges. Two vertices connected by an edge are called adjacent. The degree of a vertex is the number of edges connected to the vertex.*

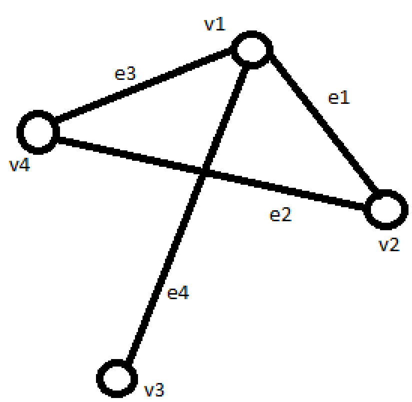

Figure 3: Simple Graph

This is a graph, G. Our V is $V = (v1, v2, v3, v4)$. Our E is $E = (e1, e2, e3, e4)$. The vertex, $v1$, has a degree of 3.

Definition 2.2. *A graph is planar if it can be embedded in a plane such that no two edges meet except at a vertex.*

Definition 2.3. *A map M consists of a planar graph G with an embedding of G in the plane.*

Definition 2.4. *A proper coloring is an assignment of colors to the regions of a graph such that no two adjacent regions are the same color.*

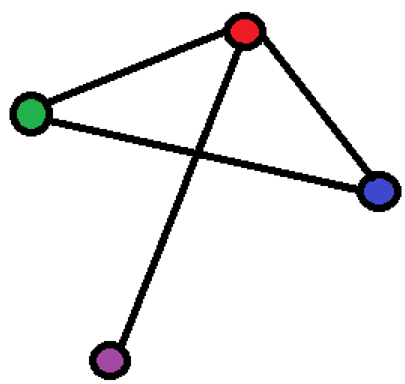

Figure 4: Coloring Example

This is a proper coloring of a graph, G, with our regions being our vertices.

Definition 2.5. *A dual graph, denoted as $D(G)$, of a graph G is a graph that has a vertex for every region of G.*

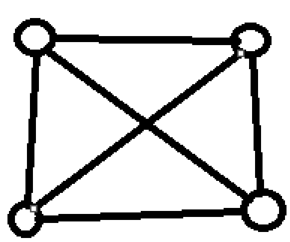

Figure 5: Graph G

This is a graph, G.

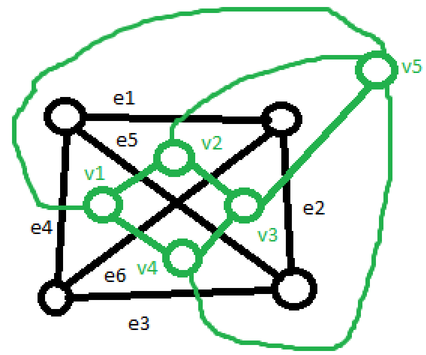

Figure 6: Dual Graph $D(G)$

This is the dual graph, $D(G)$.

To find the dual graph of a graph, each region now receives its own vertex. The new vertex is bounded by the edges of the region it is placed in. Then, to determine how many edges the new vertex receives, look back on the original graph. The new vertex receives the number of edges the vertex is surrounded by. Thus, looking at the dual graph's $v1$ of the dual graph, it is surrounded by the graph's edges $(e4, e5, e6)$ meaning $v1$ of the dual graph needs three edges also. Each new edge of the dual graph goes to the adjacent vertex of the dual graph including the outside vertex.

2.3 History of the Four Color Theorem

The Four Color Theorem was first proposed in 1852 by Francis Gutherie. Gutherie discovered this problem when he was trying to color the map of the English countries. That same year, the proposition was brought to Augustus De Morgan. De Morgan introduced this proposition to several mathematicians and in 1878, Arthur Cayley asked if the Fur Color Conjecture had been solved. In the year following, Cayley published a paper entitled *On the Coloring of Maps* in which he describes the difficulties of the conjecture.

Also in that same year, 1879, Arthur Kempe announced he had a proof to the conjecture.

This was the first time the Four Color Conjecture became known as the Four Color Theorem. In Kempe's proof, he used a method we now refer to as Kempe Chains. But Kempe's proof had a flaw that he was not able to correct. The correction came in 1891 by Percy Heawood. Heawood is credited with solving the Five Color Theorem.

In 1880, Peter Tait attempted to solve the Four Color Conjecture with two failed proofs, but this was the first time graphs were used in an attempt to solve the Four Color Conjecture. Tiat's first proof included using a dual graph and creating a triangular dual graph. From these graphs, he began lettering the vertices such that no two adjacent vertices were the same letter. The problem with this proof was Tait went on to add extra vertices which made his triangular dual graphs into square regions. Also, when writing his proof, he was unclear in his wording so while his method works, mathematicians are still uncertain on why it works because his rules are unclear. His second proof also came out in 1880 and Tait instead wanted to color the edges.

In 1898, Heawood contributed to the Four Color Conjecture with the help of Tait's second proof. Then, in 1904, G. D. Birkhoff introduced reducibility which later proofs relied on.

In 1976, Appel and Haken proved the Four Color Theorem using Kempe Chains and the help of computers. This was the first proof of its kind which caused controversy through the mathematics community. It took until about 2005 for the skepticism of the proof to dissipate.

3 12 Regions

In Thomas L. Saaty's article, *Thirteen Colorful Variations on Gutherie's Four-Color Conjecture*, on page seven, there is the following sentence:

Conjecture 3.1. *"A map of less than 12 regions has at least one regions bounded by less than 5 edges" [5].*

The following sections will explore looking at both a graph, G and its dual graph, $D(G)$.

Vertices will be the regions in the following graphs. The following graph has ten vertices.

3.1 Graph G

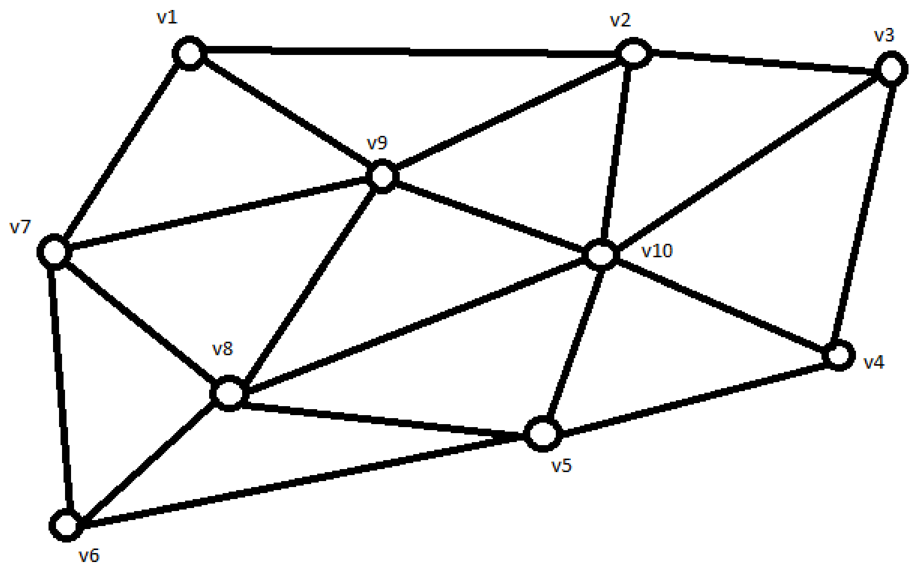

Figure 7: Graph with 10 Regions

Of these ten vertices, $(v1, v2, v3, v4, v5, v6, v7)$ all are bounded by less than five edges. While $v8$ is bounded by five edges and $v9$ and $v10$ are bounded by six edges.

This is because if we assume any graph of at most 12 regions is four-colorable, then every region of the graph must be bounded by at least five edges. But when looking at the previous graph, we can see that eight regions are bounded by less than five edges.

To color the regions of this graph with only four colors, either $v9$ or $v10$ will be the starting regions and will not be colored until the end. In the following example, $v10$ is the starting region. To color the graph, assign the first color to the vertices such that any adjacent vertices are not the same color. Then repeat this process until the only region not colored is the starting region. Thus, once all other regions are colored, the starting region will only have one coloring choice. In this example, the starting region was colored with the third color.

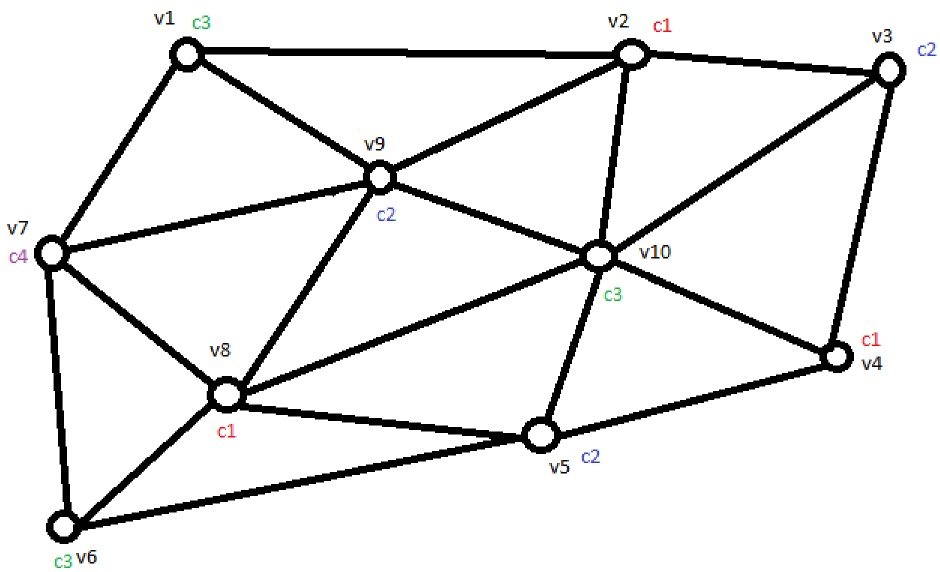

Figure 8: Graph with 10 Regions Colored

So, the graph is four-colorable with less than 12 regions and having eight regions being bounded by less than five edges.

3.2 Dual Graph $D(G)$

Another way to look at this problem is the use of the dual graph. Below is the dual graph before removing the first graph, G.

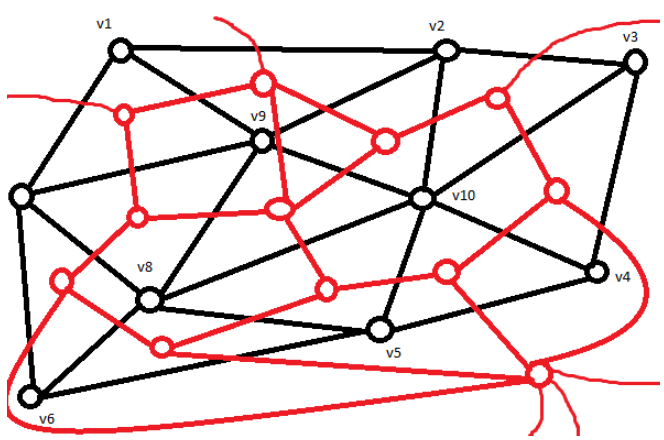

Figure 9: Dual Graph of Graph with 10 Regions

Now, once we remove our graph, G, and label our new vertices, we have the following dual graph, $D(G)$. The following dual graph has twelve vertices.

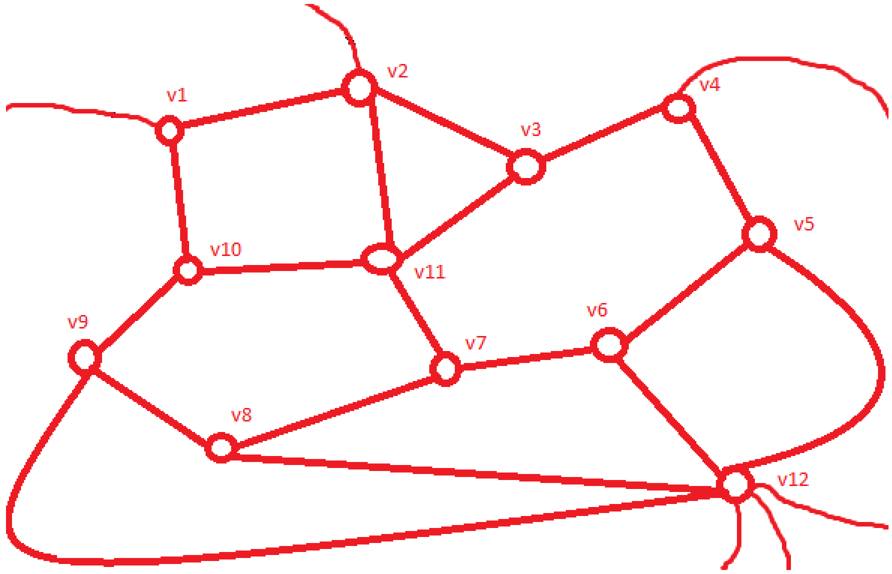

Figure 10: Dual Graph Labeled

Now, $(v1, v2, v3, v4, v5, v6, v7, v8, v9, v10, v11)$ all are bounded by less than five edges. But $v12$ is bounded by six edges. To color the dual graph, $D(G)$, the starting vertex will be $v11$ because the vertex is bounded by the most number of edges. The following dual graph was colored in the same way as the graph, G, above.

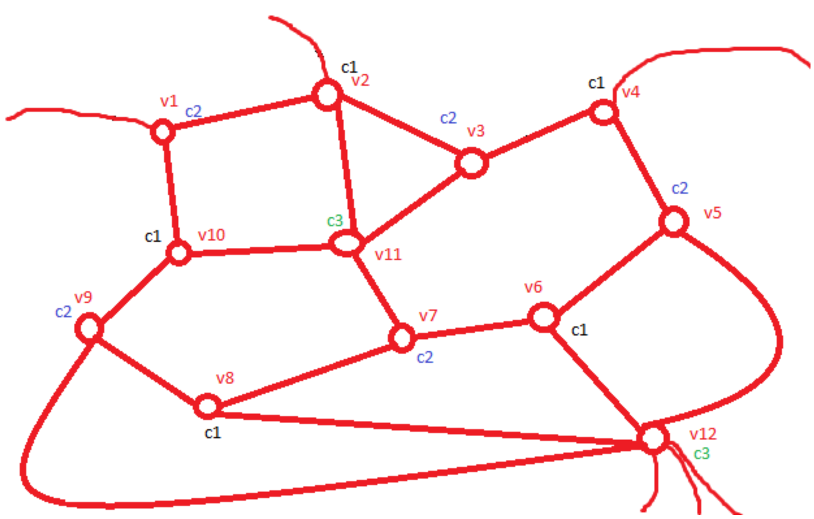

Figure 11: Dual Graph Colored

But the dual graph is only colored in three colors. Let's try to look at this mathematically to understand.

3.3 Some Mathematics

Any graph of at most 12 regions has at least one region with no more than four edges. Then, suppose the graph has n vertices, m edges, and r regions and using Euler's formula with substitution, it gives

$$n + r = 2 + m.$$

Now, assume the graph has no regions of degree one or two. Then,

$$3n \leq 2m.$$

Then, assuming every region of the four-colorable graph is bounded by at least five edges,

$$5r \leq 2m.$$

Substituting $5r \leq 2m$ into Euler's formula, $m \geq 30$. Also, substituting $3n \leq 2m$ into Euler's formula, $m \leq 3r - 6$. [5]

Then, plugging 12 in for r,

$$m \leq 3(12) - 6$$
$$m \leq 36 - 6$$
$$m \leq 30.$$

This contradicts with the earlier statement of $m \geq 30$ because m cannot be both greater than or equal to and also less than or equal to 30. Since the conjecture is a graph of less than 12 regions has at least one region bounded by less than five edges, then having $r = 12$ be plugged into $5r \leq 2m$, it shows that

$$5r \leq 2m$$
$$5(12) \leq 2m$$

$$60 \leq 2m$$

$$30 \leq m.$$

Thus, a graph of 12 regions will have at least one region bounded by less than five edges.

4 Conclusion

In conclusion, a reason why if we have a graph of less than 12 regions, at least one region is bounded by less than five edges is because when assuming the a four-colorable graph is bounded in all regions by five edges, a contradiction was formed.

This contradiction was formed because it is not possible to be both greater than and less than a number. In the proof, m was shown as being less than or equal to 30 after it was already shown to be greater than or equal to. When using a number less than 12, it was shown that m was proven to be true.

Coloring is an important part of learning about graphs and understanding why if a graph has a region bounded by any number, how to implement a coloring that works. The Four Color Theorem is a difficult theorem to prove by hand and while it has been solved, it is still a hard topic to wrap one's mind around.

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