N-QUEENS PROBLEM

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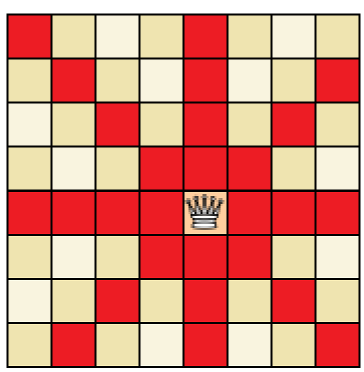
Abstract

Using combinatorics in this paper, we will discuss three different methods in solving the n-queens problem. We will find the maximum and minimum number of queens we can place on an $n \times n$ chessboard. Also, we will use latin squares, latin rectangles and circulant matrices as another method of placing the queens on a chessboard.

1 Introduction

The board game chess consists of six unique game pieces and an 8×8 chessboard. The pieces include knights, rooks, bishops, pawns, kings and queens. Each game piece has its own distinct movement on the chessboard. Many of these pieces have been worked into different problems in the field of combinatorics. Some examples of these problems include the knights tour, rook polynomials, and the queens problem. The main purpose of this paper is to discuss the different methods behind the n-queens problem. First, we need to understand what the n-queens problem is. The n-queens problem asks how many queens can be placed on a chessboard where no two queens can attack each other but every other spot on the board can be attacked. To achieve this we must know that a queen can attack any piece that lay on her row, column, or diagonals.



There is some history that brought the n-queens problem to light. The problem of the eight queens was first posed in 1848 by M. Bezzel, a German chess player. Bezzel came up with the original idea of the eight queens problem but did not actually find any solutions to it. In 1850 this exact problem was studied by C.F. Gauss. Gauss seems to be known for originating this problem and being the first to solve it but that is really not the case. In reality, F. Nauck was first to solve the problem in 1850 because he found all 92 solutions whereas Gauss only found 72 of the solutions. T.B. Sprague gave the complete number of solutions for $n = 4, 5, ..., 11$ as shown below. [1]

n	Solutions
4	2
5	10
6	4
7	40
8	92
9	352
10	724
11	2680

[1]

The earliest we can find any information on the n-queens problem was back in 1869 by F.J.E Lionnet, who wrote a general paper explaining this problem. However, it wasn't until 1874 that the first proof was discovered. E. Pauls wrote the proof on how n non-attacking queens can be positioned on an $n \times n$ chessboard but W. Aherns is normally claimed to be the first person to have proven this. Aherns has actually written in his work that he has gotten his methods from Pauls so this might be another historical error in the n-queens problem timeline. [1]

In this paper, we will discuss the different methods that exist for solving the n-queens problem. We will start off by explaining an application using latin squares, latin rectangles, and 2-circulants which will be defined in the next section. Then, we will go on to discuss the Spencer-Cockayne Construction and Welch's method.

2 Definitions

In this section, a few definitions are going to given to help in understanding the rest of the paper. A **latin rectangle** is an $M \times N$ matrix that includes integers $1, 2, 3, ..., N$ where the integers appear no more than once in each row or column. A **latin square** is an $N \times N$ matrix that includes integers $1, 2, 3, ..., N$ where the integers appear no more than once in each row or column. Figure 1 will show examples of a latin rectangle and latin square. [2]

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 6 & 4 & 5 \\ 3 & 4 & 2 & 5 & 6 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 5 & 1 & 4 \\ 3 & 5 & 4 & 2 & 1 \\ 4 & 1 & 2 & 5 & 3 \\ 5 & 4 & 1 & 3 & 2 \end{bmatrix}$$

Figure 1: A is a 3×6 latin rectangle and B is a 5×5 latin square.

Much like a latin square, a **circulant matrix**, denoted by $circ(c_1, c_2, ..., c_n)$, is an $N \times N$ matrix that has the first row include integers $1, 2, 3, ..., N$ and every row after that has to move each element one column to the right from the row previous to it. A **g-circulant**, denoted by $g - circ(c_1, c_2, ..., c_n)$, is an $n \times n$ matrix where the first row includes integers $1, 2, ..., n$ and each row after that has to move elements g columns to the right from the row previous to it. An example of a circulant matrix and a **2-circulant matrix**, which is a matrix that moves elements two columns to the right from the previous row, will be given in figure 2 below. [2]

$$C = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \\ 4 & 5 & 1 & 2 & 3 \\ 3 & 4 & 5 & 1 & 2 \\ 2 & 3 & 4 & 5 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 2 & 3 \\ 2 & 3 & 4 & 5 & 1 \\ 5 & 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 1 & 2 \end{bmatrix}$$

Figure 2: C is a 5×5 circulant matrix and D is a 5×5 2-circulant matrix.

3 Methods Solving the N-Queens Problem

In this section, we will discuss three different methods in solving for the n-queens problem. Each method will be explained in depth and will be provided with multiple examples using smaller sized chessboards.

Solution Using 2-Circulants

In the next section, we will introduce a method using 2-circulants, latin rectangles, and latin squares that will help solve the n-queens problem. We will find solutions for all even and odd n for $N \geq 4$ using this 2-circulants method. The even n will be solved using two constructions. The first construction will consist of solving for $N = 6\alpha - 2$ and $N = 6\alpha$ for $\alpha = 1, 2, 3, ..., n$. An example for this construction includes letting $\alpha = 1$. That means we could use this solution to solve for a 4×4 and 6×6 chessboard if we plug in one for α in the equations. The second construction will solve for $N = 6\alpha - 2$ and $N = 6\alpha + 2$ for $\alpha = 1, 2, 3, ..., n$. An example for this construction involves letting $\alpha = 1$. We would be able to solve for a 4×4 and 8×8 chessboard. These constructions will work for whatever α we choose. The odd n will also be solved using two constructions. The first construction for odd n will solve for $N = 6\alpha - 1$ and $N = 6\alpha + 1$ for $\alpha = 1, 2, 3, ..., n$. Using the same example as above, if we let $\alpha = 1$, we can solve for a 5×5 and 7×7 chessboard. The second construction will solve for $N = 6\alpha - 1$ and $N = 6\alpha + 3$ for $\alpha = 1, 2, 3, ..., n$. Also, letting $\alpha = 1$ using this construction, we can solve for a 5×5 and 9×9 chessboard. We can use these constructions using any α. To help understand the upcoming constructions we have to introduce a lemma.

Lemma 1. *Let G=$g - circ(c_1, c_2, ..., c_n)$ where there exists positive integers r and N. Then let $M = \frac{N}{gcd(N,r)}$.There are two rule that go along with this lemma.*

1) If we have a $gcd(N, r) = 1$, G will be considered a latin square.

2) If we have $k = gcd(N, r) \neq 1$, G will be constructed of k identical latin rectangles.

Proof. Assume $c_j \in G$ where $1 \leq j \leq N$. c_j is defined by $g(i-1)+j(modN)$ where $i = 1, 2, ..., N$ for every row i.

1) Let the $gcd(N, r) = 1$. This means $M = \frac{N}{1}$. So, $M = N$. This would make an $N \times N$ matrix where all columns are unique.

2) Let $k \neq 1$. If this is true we will have

$$i + \frac{N}{g}, i + 2\frac{N}{g}, ..., i + (k-1)\frac{N}{g}$$

for every row i. This shows that there are identical columns and the the columns are unique making k identical latin rectangles. □

Construction 1: Solutions for Even N

Introducing this method from C. Erbas and M. Tanik construction 1 will find the solutions for $N = 6\alpha - 2$ and $N = 6\alpha$ when $\alpha = 1, 2, 3, ...$ using only four steps. The goal of these steps is to create a latin square and then correlate it to an $n \times n$ chessboard. Throughout this section we will use a specific example of a 6×6 chessboard to help understand the steps of the construction. [2] **Step 1**: Create a 2-circulant $\frac{N}{2} \times N$ latin rectangle, denoted $L_1(N)$, starting the first row with $1, 2, 3, ..., N$.

$$L_1(6) = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 6 & 1 & 2 \end{bmatrix}$$

Since this latin rectangle is a 2-circulant $g = 2$. The $gcd(N, 2) = 2$. Since the $gcd(N, 2) \neq 1$, we will obtain a latin rectangle. For this example, $L_1(6)$ was created above by letting $N = 6$. When $N = 6$ we receive a $gcd(6, 2) = 2$. So we will need to use part two of our lemma. We will acquire a latin rectangle of the order $\frac{6}{2} \times 6$, meaning we will receive a latin rectangle of order 3×6 . $L_1(6)$ is definitely a 2-circulant matrix because each element in the row moves two columns to the right from the row previous to it.

Step 2: Create a second 2-circulant $\frac{N}{2} \times N$ latin rectangle, denoted $L_2(N)$, starting the first row with $2, 3, ..., N, 1$.

$$L_2(6) = \begin{bmatrix} 2 & 3 & 4 & 5 & 6 & 1 \\ 6 & 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 6 & 1 & 2 & 3 \end{bmatrix}$$

Using the same reasoning as the first step, the $gcd(6, 2) = 2$. So, since the $gcd(6, 2) \neq 1$ we will receive a latin square of order $\frac{N}{2} \times N$. $L_2(6)$ is a 3×6 latin rectangle and has the first row containing elements $2, 3, 4, 5, 6, 1$. Each row after that has moved elements two places to the right from the row previous to it, making it a 2-circulant matrix.

Step 3: Place $L_1(N)$ over $L_2(N)$ to create a latin square of order N. Denote this latin square as $L(N)$.

$$L(6) = \frac{L_1(6)}{L_2(6)} = \left[\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 6 & 1 & 2 \\ \hline 2 & 3 & 4 & 5 & 6 & 1 \\ 6 & 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 6 & 1 & 2 & 3 \end{array}\right]$$

This creates a latin square because the first latin rectangle has the first row consist of $N = 1, 2, 3, 4, 5, 6$. The odd numbers are in the odd columns and the even numbers are in the even columns. In contrast, the second latin rectangle has a first row that consists of $N = 2, 3, 4, 5, 6, 1$ which contains the even elements in the odd columns and the odd elements in the even columns. Since the latin rectangles have opposite columns it turns out that they get perfectly placed so that no element appears more than once in each row or column. $L(6)$ makes up a latin square using the two latin rectangles that were created in the previous two steps. The first three rows are $L_1(6)$ and the last three rows are $L_2(6)$. This creates a latin square because no two elements were repeated in any row or column of this matrix.

Step 4: Wherever a two exists on the latin square created, place a queen corresponding to that spot on an $N \times N$ chessboard.

	Q				
			Q		
					Q
Q					
		Q			
				Q	

$$\begin{bmatrix} 1 & \underline{\mathbf{2}} & 3 & 4 & 5 & 6 \\ 5 & 6 & 1 & \underline{\mathbf{2}} & 3 & 4 \\ 3 & 4 & 5 & 6 & 1 & \underline{\mathbf{2}} \\ \underline{\mathbf{2}} & 3 & 4 & 5 & 6 & 1 \\ 6 & 1 & \underline{\mathbf{2}} & 3 & 4 & 5 \\ 4 & 5 & 6 & 1 & \underline{\mathbf{2}} & 3 \end{bmatrix}$$

Theorem 1. *Let N be an integer. When $N \geq 4$ and $N = 6\alpha$ or $N = 6\alpha - 2$, for $\alpha = 1, 2, ...,$ use the steps above to create a latin square, $L(N)$, and transpose the information onto an $N \times N$ chessboard. The queens should be placed on the spots containing a 2. This will then give a solution that coincides with the N-Queens Problem.*

Proof. The queens are placed on all the two's because the first latin rectangle, $L_1(N)$, places the queens using the coordinate $(i, 2i)$ for $i = 1, 2, ..., \frac{N}{2}$. Whereas the second latin rectangle, $L_2(N)$, places the queens on the coordinate $(\frac{N}{2} + i, 2i - 1)$ for $i = 1, 2, ..., \frac{N}{2}$. Since a latin square is created, we need to make sure the positive and negative diagonals only hold one queen each. We check this by taking the coordinate (i, j) and adding $i + j = k$ where k is the positive diagonals. So, $L_1(N)$ has $k = 3i$ where $i = 1, 2, ..., \frac{N}{2}$ positive diagonals and $L_2(N)$ has $k = 3i + \frac{N}{2} - 1$ where $i = 1, 2, ..., \frac{N}{2}$ positive diagonals. Relating this information to $N = 6\alpha$, for $L_1(N)$ we would have $k = 3i = 0 (\text{mod } 3)$. For $L_2(N)$ using $N = 6\alpha - 2$ we would have $k = 3(\alpha + i) - 2 = 1 (\text{mod } 3)$. Lastly, for $L_2(N)$ using $N = 6\alpha$, we would get $k = 3(\alpha + i) - 1 = 2 (\text{mod } 3)$. This shows that no queen falls on the same positive diagonal. Using the same ideology we could do the same thing for the negative diagonal. Instead of $k = i + j$, we need to let $k = i - j$. We will find out that no two queens lay on the same negative diagonal which proves, the two's are the only solution for this construction. □

Any two that exists in the latin square has been bolded. We need to relate this information back to a 6×6 chessboard. So, we see a two in the first row, second column of the latin square which means a queen will be placed in the first row, second column of the chessboard. We continue this process until every two has a queen correlating to its spot on the chessboard. This creates one solution of the queens problem for a 6×6 chessboard.

Construction 2: Solutions for Even N

Introducing a different method by Erbas and Tanik, for this construction we are finding the solutions for $N = 6\alpha - 2$ and $N = 6\alpha + 2$ when $\alpha = 1, 2, 3, \ldots$ using four steps. For this construction we have to name a new set, $D(N)$.

$$D(N) = \left\{ 6i + 3 \mid \ 0 \leq i \leq \left\lfloor \frac{\alpha - 1}{3} \right\rfloor \right\}$$

[2]

Throughout this construction we will present examples by letting $N = 10$ and using a 10×10 chessboard. [2]

Step 1: Much like step one from Construction 1, we must construct a 2-circulant latin rectangle having a first row of $1, 2, 3, \ldots, N$.

$$L_1(10) = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 9 & 10 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 8 & 9 & 10 & 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 7 & 8 & 9 & 10 & 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 1 & 2 \end{bmatrix}$$

Using Lemma 1, we know that the $gcd(10, 2) \neq 1$ so, we need to create a latin rectangle of order $\frac{N}{2} \times N$. $L_1(10)$ starts off with the first row consisting of $1, 2, \ldots, 10$. Each row after that moves two places to the right from the previous row making this latin rectangle a 2-circulant.

Step 2: Construct a 2-circulant matrix but let the first row of the matrix start off with $N - d + 1, N - d + 2, \ldots, N, 1, 2, \ldots, N - d$. This means we have to subtract the number we get for d from N to help with creating our first row. This latin rectangle will be denoted by $L_3(N, d)$.

$$L_3(10, 3) = \begin{bmatrix} 8 & 9 & 10 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 & 10 & 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 6 & 7 & 8 & 9 & 10 & 1 & 2 & 3 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 1 \\ 10 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{bmatrix}$$

For the same reason as the previous step, the $gcd(10,2) \neq 1$ so, we need to construct a latin rectangle of order $\frac{N}{2} \times N$. $L_3(10,3)$ was created because $d = 3$. The first row had to start off with $8 - 3 + 1 = 6$ and end with $10 - 3 = 7$. Then each row after that has to move each element two places to the left from the row previous to it.

Step 3: Place $L_1(N)$ over $L_3(N,d)$ to construct a latin square.This will be denoted by $L(N,d)$.

$$L(10,3) = \frac{L_1(10)}{L_3(10,3)} = \left[\begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 9 & 10 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 8 & 9 & 10 & 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 7 & 8 & 9 & 10 & 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 1 & 2 \\ \hline 8 & 9 & 10 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 & 10 & 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 6 & 7 & 8 & 9 & 10 & 1 & 2 & 3 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 1 \\ 10 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array}\right]$$

The first five rows of the latin square are made up of $L_1(10)$ and the last five rows are made up of $L_3(10,3)$. This is considered a latin square because the matrix is 10×10 making it square and no two elements are repeated in any column or row. This construction forms a latin square because all odd elements are in the odd columns in $L_1(10)$ whereas all odd elements are in the even columns of $L_3(10,3)$. Since the first two steps constructed latin rectangles, that means no element is repeated in any row or column. So, if all even numbers from $L_1(10)$ are in the same column as all odd elements from $L_3(10,3)$, no element will be repeated making it a latin square.

Step 4: We are going to relate our latin square that we found to an $N \times N$ chessboard. This step may contain multiple solutions so we have a theorem to ensure that all solutions are found. [2]

Theorem 2. *Let N be an integer. For $N \geq 4$ when $N = 6\alpha - 2$ or $N = 6\alpha + 2$, for $\alpha = 1, 2, ...$, use the steps above to create a latin square, $L(N,d)$, and transpose the information onto an $N \times N$ chessboard. Define a set $S(N,d)$ such that,*

$$S(N,d) = \{s \in Z \mid d + 1 \leq s \leq N - (2d - 2)\}.$$

The queens will then be placed where the integer s falls on the latin square. This will give solutions that correspond to the n-queens problem.

Proof. We need to check if the positive diagonals and negative diagonals of an $N \times N$ chessboard contains at most one queen. First, let assume k is a positive diagonal. This means that $2 \leq k \leq 2N$ since $L(N,d)$ uses 2-circulants. This shows us that $x_{i+1,j-1} = x_{i,j} - 3(\text{mod N})$ for the top half of $L(N,d)$. Using the same reasoning for the bottom half of $L(N,d)$ for elements of $i = \frac{N}{2}$ and $\frac{N}{2} + 1$, we get $x_{\frac{N}{2}+1,j} = x_{\frac{N}{2},j+1} - 3 - d(\text{mod N})$. Now we need to prove for negative diagonals. If k is a negative diagonal of $L(N,d)$ $1 - N \leq k \leq N - 1$. Since the top half of $L(N,d)$ is 2-circulant and contains two successive negative diagonals $x_{i,j}$ and $x_{i+1,j+1}$, we get $x_{i+1,j+1} = x_{i,j} - 1(\text{mod N})$. Using the same reasoning as above, we can find the bottom half of $L_(N, d)$. The only difference is letting $i = \frac{N}{2}$ aand $\frac{N}{2} + 1$. This proves that all the elements in the diagonals are unique. □

Relating this back to the example, let $N = 10$ and $d = 3$.

$$\begin{aligned} S(10,3) &= \{s \in Z \mid 3+1 \leq s \leq 10-(2(3)-2)\}. \\ &= \{s \in Z \mid 4 \leq s \leq 6\}. \end{aligned} \quad [2]$$

This means we can place a queen anywhere we see a 4, 5 or 6 on our latin square since s falls between 4 and 6. The solutions to the 10-queens problem will be given below.

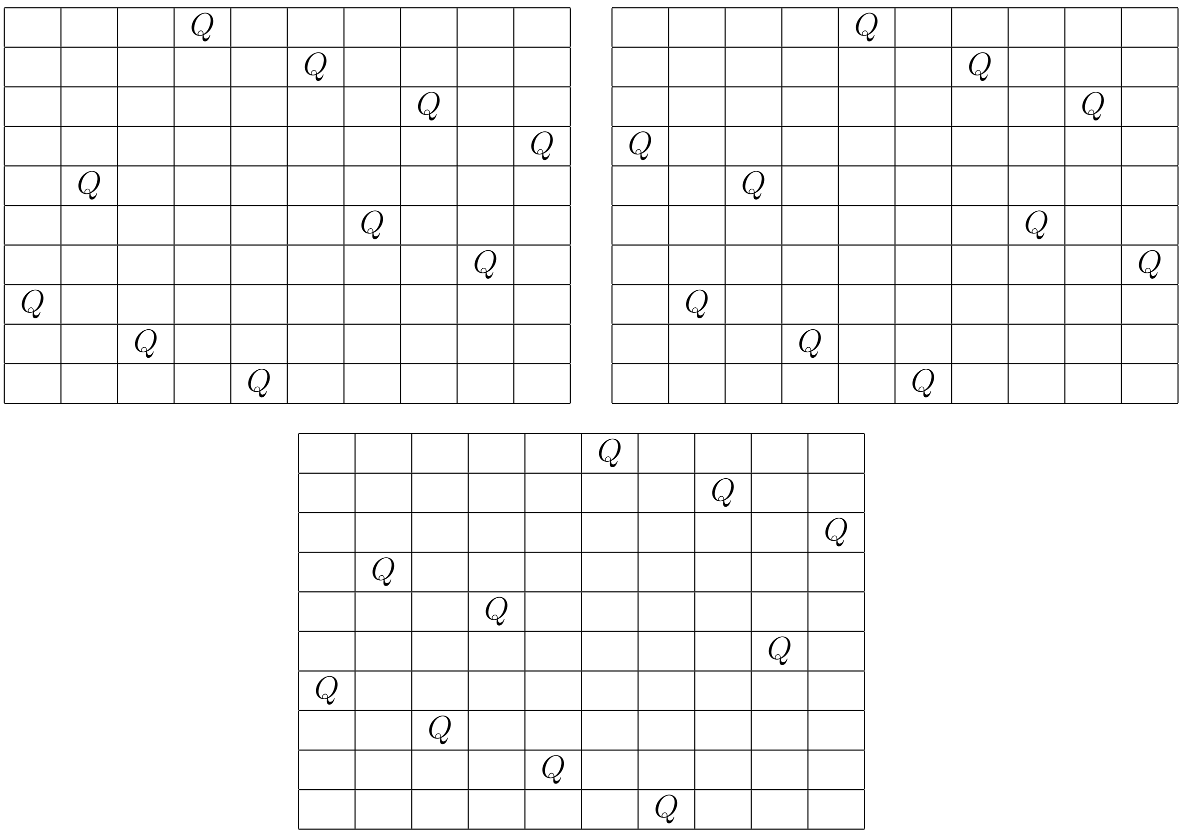


Figure 3: Solutions for the 10-queens problem using latin squares.

Construction 3: Solutions for Odd N

Construction 3 will find solutions to $N = 6\alpha - 1$ and $N = 6\alpha + 1$. There is only two steps involved in solving this construction making it simpler than the previous constructions. To help understand the steps we will use an example and let $N = 7$. [2]

$$L(7) = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 7 & 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 6 & 7 & 1 & 2 & 3 \\ 2 & 3 & 4 & 5 & 6 & 7 & 1 \\ 7 & 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 7 & 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 6 & 7 & 1 & 2 \end{bmatrix}$$

$L(7)$ is considered a 2-circulant latin square because no element is repeated in either row or column and it is a square 7×7 matrix.

Step 2: Relate the latin square to its corresponding $N \times N$ chessboard. Any number can be chosen as a solution to the n-queens problem.

Since a latin square consists of no element appearing more than once in each row or column, we will not receive attacking queens in any row or column with any number in this latin square. This means any number from one to seven will generate a solution to the 7-queens problem. We will receive seven possible solutions to the seven queens problem. We can put a queen wherever we see a one for one specific solution. We can do the same thing for two through seven. The construction will work for all multiples of six plus one and all multiples of six minus one.

Construction 4: Solutions for Odd N

This construction will generate the solutions for $N = 6\alpha - 1$ and $N = 6\alpha + 3$ when $\alpha = 1, 2, 3...$ using only four steps. This construction is going to be different than the previous ones because we are going to create a latin square, $L(N-1, d)$. Two new sets also need to be defined. When $N \geq 4d - 1$ there is a $P(N, d)$, such that

$$P(N, d) = \{p \in Z \mid \left(\frac{N-1}{2}\right) - d + 2 \leq p \leq \left(\frac{N-1}{2}\right) + 1\}. \quad [2]$$

Also, there exists a $N < 4d - 1$ that defines the set $Q(N, d)$ as,

$$Q(N, d) = \{q \in Z \mid d + 1 \leq q \leq N - (2d - 1)\}. \quad [2]$$

Step 1-3: Construct a latin square using the first three steps of the previous constructions. The latin square that needs to be created is $L(N-1, d)$.

Step 4: Decide whether $N \geq 4d - 1$ or $N < 4d - 1$. This will help with deciding which set to use. The solutions will be found using the sets above, then we will relate the latin square we created to an $N \times N$ chessboard based off of the solutions found in the specific sets.

Together all four of these constructions show the solutions to all N for the n-queens problem using latin squares, latin rectangles, and 2-circulants. Now we are going to explain another way to solve the n-queens problem. This method is called "The Spencer-Cockayne Construction".

The Spencer-Cockayne Construction

Lets state that A **domination number** is the minimum number of queens placed on an $N \times N$ chessboard. The goal of this particular construction is to give the domination numbers of each $N \times N$ chessboard. The Spencer-Cockayne Construction starts out by placing a single queen directly in the middle of the chessboard. Using an example of a 5×5 chessboard, we can place one queen right in the middle of the board. We now create a 3×3 buffer around the single queen. The queen will be able to kill anything placed in this area. In the example below, we can see that this 3×3 method Spencer uses works well but we still have eight spots left open where the queen cannot attack. We represent the queen here by Q and the open spots by O. [3]

Figure 4: 5×5 Chessboard using Spencer-Cockayne Construction

	O		O	
O				O
		Q		
O				O
	O		O	

[3]

A knights move in chess consists of moving up two, over one or over two, up one. Paying attention to the example above, we notice that each open spot is exactly a knights move away from the queen. To actually solve this 5×5 chessboard problem, we need to place four queens in four of the O squares. Since the queens can not attack the places with the O, we will need to place another queen to help attack those open places. After placing one queen in one of the open spots, we will have three other spot open. These last three spots will only be covered if we place a queen on all of them because they do not lay on any row, column or diagonal the other queens are on. The queens we added to solve this problem are bolded below. [3]

Figure 5: 5×5 Chessboard Solution

			Q	
Q				
		Q		
				Q
	Q			

[3]

Similarly, we can use the exact chessboard from above to solve a 9×9 chessboard. If we can solve a 9×9 chessboard using this method, that means we can solve an 8×8 chessboard, since it is included in the 9×9 board. As a side note, we can come to the conclusion that the domination number for this 9×9 chessboard is 5. This can be denoted by $dom(figure\ 5) = 5$. [3]

Figure 6: 9×9 Chessboard Solutions

					Q			
		Q						
				Q				
						Q		
			Q					

[3]

This particular method continues to add four queens at a time when trying to control larger square chessboards. This then brings us to an 11×11 chessboard. All 5 queens on the 9×9 chessboard can cover every space up until it hits an 11×11 chessboard. Then, we will have eight uncovered spaces using only five queens. This is when we use Spencer's method again and add four more queens. We place them identical to the way we placed the first set of four queens on the 5×5 chessboard. Now, these nine queens will succeed in covering every space on an 11×11 chessboard through a 15×15 chessboard. It has not been concluded that the least amount of queens you can place on a 15×15 chessboard is exactly nine. If we complete this method again, we will place thirteen queens on a 16×16 chessboard and these queens can cover through to a 21×21 chessboard. However, it has been proven that fewer than thirteen queens can achieve the goal of attacking every square on the 21×21 board. [3]

Upper Bound and Lower Bound of Domination Numbers

On the topic of domination numbers, the upper bound was founded by Welch and the lower bound was founded by Spencer. They both came up with their own theorems on where the least amount of queens can be placed on a chessboard. Welch discovered that the upper bound was roughly two thirds. He created this theorem by taking a $3m \times 3m$ chessboard and dividing it into nine $m \times m$ squares. He then continued to place m queens in the lower left hand and upper right hand corners of the board. His theorem is written below. [3]

Theorem 3. *The upper bound of domination numbers is*

$$For \;\; n = 3m - r, 0 \leq r < 3 :$$
$$\gamma(Q_{n \times n}) \leq 2m + r.$$

Proof. We must prove this theorem by stating that we are using a $3m \times 3m$ chessboard and we need to divide it into nine $m \times m$ squares. Next, we need to place m queens in the upper right hand and lower left hand of the $3m \times 3m$ chessboard. After placing the m number of queens, we will have r rows and columns that are left over which can be solved by placing r number of queens. So, this tells us that roughly about $\frac{2}{3}$ of queens are needed which proves Welch's upper bound of domination numbers theorem. □

Spencer on the other hand founded the lower bound of the domination number. His discovery told us that the minimum number of queens we can place on an $n \times n$ chessboard is about one half of the number of queens. We can show a few examples to justify this information a little more. First, think of a 2×2 chessboard. If you want to cover every square on that board all you have to do is place one queen. Half of two is one which proves this to be true. Second, think about a 10×10 chessboard. As we said above the minimum number of queens we can place on this board is five. Half of 10 is five which is another example to help justify this theorem. [3]

Theorem 4. *The lower bound for domination numbers is*

$$\gamma(Q_{n\times n}) \geq \frac{1}{2}(n-1).$$

Proof. Assume that there is a minimum number of queens that can be placed on an $N \times N$ chessboard. We do this by letting $\gamma = \gamma(Q_{N\times N})$. Now, let $\gamma \leq N-2$. This means two rows must be empty on our chessboard. So, let x be the left most open column and let y be the right most open column. Also, let w be the lowest empty row possible and let z be the highest empty row possible. Now lets assume $x - y \geq z - w$. $z - w$ will be contained in $x - y$ so that we get $2(x - y)$. Since z is the highest empty row and w is the lowest empty row, we can let $(w-1) + (n - (w + (x-y))) = n - x - y - 1$. Now we need to make sure there are γ many queens on the chessboard. So, since there is $2(x-y)$ squares, we get $2(n-(x-y)-1)+4(\gamma-(n-(x-y)-1)) \geq 2(x-y)$. USing simply algebra and by solving for γ, we can see $\gamma \geq \frac{1}{2}(n-1)$ which is Spencer's lower bound thus ending the proof. □

Both of these theorems successfully helped narrow down the minimum number of queens that can be placed on a chessboard.

4 Conclusion

To conclude, the n-queens problem is an intriguing topic in the mathematical field of combinatorics. This paper consisted of three methods in finding the solutions to the n-queens problem. Even though this paper holds a great deal of information on the n-queens problem this was only three possible methods. There are plenty of other theorems and methods out there still waiting to be discussed and discovered. Some examples of these new methods include solving the n-queens problem by computer rather than by hand. That is an intriguing concept and may be looked at in more detail later on.

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