TOPSPIN: OVAL-TRACK PUZZLE
TAKING APART THE TOPSPIN ONE TILE AT A TIME

Honors Thesis

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Thesis Abstract

A Topspin "Oval Track" puzzle consists of 20 numbered tiles in an oval-shaped track and a flipping window that reverses the 4 tiles in the window. The solvability of the puzzle uses permutations which are combinations where the order matters. A puzzle is considered solvable if each permutation in S_n can be mapped to a spot in the original position through the three different moves the puzzle can make; a left shift, a right shift, and the flip which reverses the order of the 4 tiles in the window. I wanted to find out what math was involved in solving this puzzle. I had certain topics that I wanted to find out more information about, but the major question I had was "what is the fewest number of moves it takes to solve a puzzle". Other topics I had were what made a puzzle unsolvable and what other types of puzzles use the same kind of math to solve them. To construct this research, I had read different scholarly articles that talked about the Topspin as well as physically looked at the puzzle and see how it works. I had found that there was no way to determine the fewest number of moves it takes to solve a puzzle since it's impossible to decide what a "more scrambled" puzzle is compared to another. One scrambled puzzle might look completely different from another, and still have the same number of moves to solve it. In addition to this, I was also able to find that the Rubik's Cube is solved like the Topspin.

Introduction

Puzzles are a great way to get your mind going and different puzzles have different advantages for your brain. In addition to being good for your brain, puzzles are also fun. With being a math major, we have all this knowledge that can be used to solve different puzzles from the past and even ones we find in the future. The puzzle that will be “taken apart” in this paper is the Topspin Oval-Track Puzzle. When naming a puzzle, we use the notation

$$[t, n]$$

where n is the total number of tiles, and t is the number of tiles in the turnstile. The typical Topspin is a [4, 20] puzzle.

	19	20	**1**	**2**	**3**	**4**	5	6	
18									7
17									8
	16	15	14	13	12	11	10	9	

Movements of a Topspin

There are three basic moves that can be performed on a Topspin Puzzle. They are the left shift (counterclockwise), the right shift (clockwise), and the turnstile flip. The left and right shifts are inverses of each other so if you perform one then the other, you’ll be back to where you started. The inverse of the turnstile flip; however, is the flip itself. If you perform the flip twice, you are at the same spot as before the flips.

Goals of This Paper

Throughout this paper, we will be talking about the solvability of a Topspin, different examples of solvable and unsolvable puzzles, and how the Topspin relates to other puzzles of the same classification.

1 Definitions

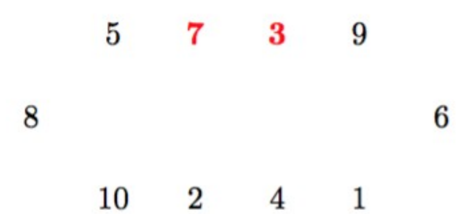
There are certain terms and concepts that will be talked about throughout the paper, so here are some overviews of them.

1.1 Permutations

A permutation is a way in which a set of numbers or things can be ordered. You can think of it as a combination, but in this case, the order of the numbers matter. For this puzzle, the numbered tiles can be arranged in a certain order. All of these different orderings are called permutations. Let's call the ascending order of the [4,20] puzzle the identity permutation. There are actually 20! different permutations that a [4,20] puzzle has, which comes out to being over 2 quintillion permutations. We can represent a permutation σ by the sequence σ[1], σ[2], ..., σ[n].

1.1.1 Cycle Notation

Let's look at a smaller example of a Topspin Puzzle. This is a [2, 10] puzzle.



As we had seen above, we can write a

permutation as a sequence. The sequence for this permutation would look as such:

σ[1] = 7, σ[2] = 3, σ[3] = 9, σ[4] = 6, σ[5] = 1, σ[6] = 4, σ[7] = 2, σ[8] = 10, σ[9] = 8, σ[10] = 5

There are other ways that we can write or express our permutations. One is in array form, which looks like this:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 7 & 3 & 9 & 6 & 1 & 4 & 2 & 10 & 8 & 5 \end{pmatrix}$$

The top row corresponds to the starting positions of the puzzle, whereas the bottom row corresponds to what numbers in the permutation are actually in those positions. So tile 7 would be in the 1 position, tile 3 would be in the 2 position, and so on.

Another form is called cycle form, which looks like this:

(1 7 2 3 9 8 10 5) (4 6)

With cycle form, we always start with 1 which corresponds to the leftmost position of the turnstile. The number that comes after the 1 would be what tile is in the first position. Therefore, in the first position, we have tile 7. Next, the number after the 7 will tell us what tile is in the seventh position. In this case, tile 2 is in the seventh position. Continuing with these steps, we get to a closed parentheses after where we would have a number for which tile is in the fifth position. This is because in the fifth position, we have tile 1 which is the number we started with. After this, we need to go to the next number that we haven't used yet, and in this case, the number is 4. Since the 4 is the first number in this cycle, we know we are talking about the fourth position. We can see that the number 6 is after this, so we know that tile 6 is in the fourth position. Since the parentheses are closed after the 6, this means that it cycles back to the beginning, which tells us that tile 4 is in the sixth position. With that being said, we are able to figure out the configuration of a Topspin based on the cycle form of the permutation.

1.1.2 Permutations of the Movements of the Topspin

We had talked earlier about how there were three different moves of a Topspin. We can write permutations to represent the order of the tiles once the move is performed. Let the left shift be denoted as α, the right shift as β, and the turnstile flip as δ.

$$\alpha = (1, 2, 3, 4, ..., n-1, n)$$

$$\beta = \alpha^{-1}$$

$$\delta = (1, t)(2, t-1)(3, t-2)...([t/2], [t/2]+1) \qquad [6]$$

What we can take away from this is that when the left shift is performed, the new permutation is just the identity shifted over one unit to the left. As we can imagine, the right shift is just the identity shifted one unit to the right. With the typical [4, 20] puzzle, when we perform the flip, tiles 1 and 4 switch and tiles 2 and 3 switch. Therefore, the permutation for it would be (1, 4)(2, 3).

1.2 Transpositions

"A transposition is a permutation σ that fixes all but two elements of 1,2,...,n and switches those two." [2] All permutations can be written as a product of transpositions. They are usually written in cycle notation with length two. Our permutation for the turnstile flip is written as a product of transpositions. Let’s look at an example of how this would work in a Topspin.

Since we said that all permutations can be written as a product of transpositions, let’s use the [2,10] puzzle example from above. There are actually many ways to write a permutation with transpositions, but we will just look at one for now.

$$(1\ \ 5)\,(1\ \ 10)\,(1\ \ 8)\,(1\ \ 9)\,(1\ \ 3)\,(1\ \ 2)\,(1\ \ 7)\,(4\ \ 6)$$

In this form, if we were to multiply these out, we would get back the permutation we were working with before. We can see that there is an even number of transpositions in this permutation. We will look at the importance of this a little later.

Now let’s look at how transpositions work. Let’s say that we wanted to switch the tiles 7 and 4. We are hoping to have our new permutation look like this:

	5	**4**	**3**	9	
8					6
	10	2	7	1	

In our Topspin puzzle, we wouldn’t be able to physically pick up the two tiles and switch them. So in order to ”switch” the tiles, we would have to perform a number of moves to get the tiles in the desired positions.

1.3 Inversions

We can define an inversion of σ as a pair of numbers i and j such that $i < j$ and that $\sigma[i] > \sigma[j]$. We will look at how this works for the Topspin. If we have the following four tiles in the turnstile in this order,

we can see that tile 3 comes before tile 1. Normally $1 < 3$, but in this permutation, tile 1 is in a greater position than tile 3. Tile 1 is in position three which is greater than tile 3 being in position one.

Inversions are written in pairs and can help figure out how many swaps are needed to get back to our identity permutation. If we had a permutation $\sigma[1] = 6$, $\sigma[2] = 5$, $\sigma[3] = 7$, $\sigma[4] = 3$, $\sigma[5] = 8$, there would be four inversions:

$$\sigma[3]\sigma[4], \quad \sigma[2]\sigma[4], \quad \sigma[1]\sigma[4], \quad \sigma[1]\sigma[2]$$

If we used the permutation of our [2,10] puzzle as an example, we would have 22 pairs of inversions. This also tells us that we would need to have 22 consecutive element swaps in order to put this puzzle back to it's identity permutation.

2 Classification of Puzzles

A Topspin is considered a permutation puzzle. These types of puzzles consist of "pieces that were rearranged, or permuted, in some way, and the goal is to try to restore the pieces to their original positions" or some other desired configuration [4]. A permutation puzzle satisfies four properties.

1. Every move made will result in a new permutation of the puzzle.

2. If a permutation can be reached by a different set of moves, it is still the same permutation. In other words, a permutation can be written in different ways and still be the same permutation.

3. Each move must be invertible, which means that each move must be able to be undone.

4. If M_1 and M_2 are moves that correspond to permutations τ_1 and τ_2, then $M_1 \cdot M_2$, which represents performing M_1 followed by M_2, is either an illegal move or is the permutation $\tau_1 \cdot \tau_2$ [4].

"End Game" of the Topspin

The point of this puzzle is to put all the numbers in a certain order. In the typical puzzle, you are trying to get these numbers in ascending order. Other possible configurations could be descending order or even odds then evens. In reality, you could try to order this puzzle in the most random configuration you can think of, you just have to know how to do it. This is where we can look at the solvability of a Topspin puzzle.

3.1 Solvability

A puzzle is considered solvable if we can generate all the different permutations of the puzzle. In simpler terms, if we can put this puzzle into every different order that it can be in, then it's solvable. From this we can conclude that if we cannot generate all of these permutations, the a puzzle is unsolvable.

3.2 Symmetric Group

The group of all permutations of the set [n] = 1, 2, ..., n is the symmetric group denoted by S_n. The symmetric group of our [4,20] puzzle can be written as S_{20}. We can prove that this group of permutations is in fact a group. From our knowledge of abstract algebra, we know that in order for something to be a group, it needs to satisfy four properties.

1. Closure - If A and B are elements in G, then AB must be an element in G.
2. Associativity - For A, B, C $\in$ G, A(BC) = (AB)C.
3. Identity - There is an identity element I such that AI = IA = A for every element A$\in$G.
4. Inverse - There must be an inverse of each element. Therefore, for every A $\in$ G there exists an element B = A^{-1} such that $AA^{-1} = A^{-1}A = I$. [5]

Let α (our left shift), β (our right shift), and δ (our turnstile flip) be the permutations of the moves of the Topspin in S_n. If we take two of the permutations, α and δ, and we perform one after the other, we get back another permutation in S_n, therefore we have shown closure.

For α, β, δ $\in$ G, α(δβ) = (αδ)β. This is true because when we perform α followed by δ and β, it is the same thing as performing α and δ followed by β. Therefore, we have shown associativity. Even though S_n is associative, that doesn't mean that it's always commutative. Applying α then β works the same way as applying β then α, but a turnstile flip then α gives a different permutation than performing α then a turnstile flip.

Let α and e be permutations in S_n. The permutation e can be expressed as:

$$e = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ 1 & 2 & 3 & \dots & n \end{pmatrix}$$

When any permutation α is performed with e such as eα or αe, you get the permutation α.

Based on the statement just made and the fact that this permutation has no inversions, we have proved that e is the identity of S_n.

We had talked earlier about how α and β are inverses of each other and that δ is it's own inverse. This means that any move that is performed can be undone. Therefore, S_n is closed under inverses and based on all of this information, S_n is a group.

3.3 Attainable Subgroup

If a puzzle is not solvable, there may be certain permutations that can be generated. These groups of permutations are generally called attainable subgroups. The attainable subgroup can be denoted by A_n. A puzzle many not be able to generate a permutation such that odd tiles are next to each other, or maybe odd tiles can only be swapped with odd tiles and even tiles can only be swapped with even tiles. The attainable subgroups can help us determine if a puzzle is unsolvable and tell us what permutations cannot be generated.

3.4 Physically Solving the Topspin

When you are trying to put the tiles in ascending order, there are two different processes to go through for two groups of numbers. Numbers 1-16 are solved one way, and numbers 17-20 are solved in another.

3.4.1 Tiles 1-16

If you have a scrambled puzzle, you start by fixing the 1 to the left of the turnstile. The asterisks in the parentheses represent the tiles in the turnstile.

$$\ldots \quad 1 \quad (\quad * \quad * \quad * \quad * \quad) \quad \ldots$$

Next, you would try to get the 2 next to the 1. To do this, you will continue to put the 2 in the right-most slot of the turnstile and flip it so it is now in the left-most slot. You will do this multiple times while still keeping the 1 on the left side of the turnstile. When this is not possible, the 2 will either be in position two of the turnstile or position three like this:

$$\ldots \quad 1 \quad (\quad * \quad 2 \quad * \quad * \quad) \quad \ldots$$
$$\ldots \quad 1 \quad (\quad * \quad * \quad 2 \quad * \quad) \quad \ldots$$

If the 2 is in the second position of the turnstile, then you would perform a flip, moving the 2 to the third position. Next you shift the 2 back to the second position and perform another flip. This brings the 2 back to the third position, but we can notice that if we shift the 2 over to the fourth position, we can perform one last flip moving the 2 to the right of the 1.

If the 2 is in the third position, the process is the same as above except we start from where we are in the third position. From that position, we shift the 2 over to the second turnstile position and perform a flip. After this, we are back in the third position, but like before, we can shift it over one unit to the right and perform a flip putting the 2 to the right of the 1. We can perform these moves until we have tiles 1-16 in order.

3.4.2 Tiles 17-20

To get the tiles 17-20 in order after tile 16, we have to perform a whole different algorithm.First, we want to flip the turnstile to make it so we have the fewest number of tiles needed to swap. If with both permutations need to have two tiles swapped, we flip it to whichever one has tile 17 closest to tile 16. In total, there are 24 different permutations for the puzzle with tiles 17-20 in the turnstile. One of them is obviously the order of which we would like them to be to solve the puzzle, but the other 23 require an algorithm to put them in our desired order. Below are 12 of those configurations. There are only 12 listed because if we have one permutation and we perform a flip, we have another permutation. So in reality, there are 12 orders of the four tiles that you can work with. Of course if one of the orders that you come across when solving isn't listed, it's actually just the reverse order of one of these.

17	18	19	20	17	20	19	18	18	19	17	20
17	18	20	19	17	20	18	19	18	20	17	19
17	19	18	20	18	17	19	20	19	18	17	20
17	19	20	18	18	17	20	19	19	17	18	20

From any of these permutations, there needs to be 1, 2, or 3 transpositions performed. As we know, we can't just transpose two tiles the way that we'd want to, so there are two different types of algorithms that you can use to accomplish this. Let's start with a permutation that needs only 1 transposition to be performed.

⑰ ⑲ ⑱ ⑳

If we start with this permutation, we can see that we just need to transpose tile 18 and tile 19. Before we can do this, we need to know what our algorithm is.

There are two different ways to interpret this algorithm. We are either trying to move a tile to the left or to the right. In reality, we could do either for this permutation. We could move tile 18 to the left, or move tile 19 to the right. Let's start by looking at moving tile 19 to the right.

The algorithm is a sequence of moves that gets repeated a total of 5 times. Here is a list of the moves to perform.

1. Move the tile you want to shift to the leftmost position of the turnstile
2. Perform a flip and shift tiles one unit to the left
3. Perform a flip and shift tiles one unit to the right
4. Perform a flip and shift tiles one unit to the left
5. Perform one last flip

After this sequence of moves, tile 19 has moved 4 units to the left and should be in this position where the tiles in the circles correspond to the tiles in the turnstile:

17 18 ⑳ ① ② ⑲ 3 4

Once you perform this process four more times, your end result will look like this:

⑰ ⑱ ⑲ ⑳

Then you will have solved the puzzle.Next, let's look at at scenario where we'd have to perform two transpositions.

⑱ ⑰ ⑳ ⑲

As we can see here, tiles 17 and 18 need to be switched, as well as tiles 19 and 20. In this case, we will move tile 18 one unit to the right and move tile 19 one unit to the left. Since we already know how to move tiles to the right, this is how the tiles would look after moving the 18 to the right:

⑰ ⑱ ⑳ ⑲

The algorithm for moving tiles one unit to the left is fairly similar to the one to move a tile one unit to the right. Here are the steps:

1. Move the tile you want to shift to the rightmost position of the turnstile
2. Perform a flip and shift tiles one unit to the right
3. Perform a flip and shift tiles one unit to the left
4. Perform a flip and shift tiles one unit to the right
5. Perform one last flip

After this is performed, when we do the process four more times, tiles 19 and 20 will have been transposed.

To solve these last four tiles, all we really needed was one of the algorithms, but it's good to know how to perform the swap the other way around. When you have an order of the last four tiles that needs three swaps, it will be a combination of swapping different tiles left or right. We now know how to solve the entire [4, 20] Topspin Puzzle.

4 Examples of Puzzles

As we have seen, the [4,20] puzzle is not the only size of Topspin we can have. After talking about the solvability of the puzzle and earlier mentioning what it means for a puzzle to be unsolvable, here are some examples of other puzzles.

4.1 Solvable Puzzles

One example of a solvable puzzle is if n is even and $t \equiv 0 (\text{mod} 4)$ or $t \equiv 2 (\text{mod} 4)$ [3]. Our [4, 20] puzzle satisfies these constraints so we know that it is a solvable puzzle, but let's take a look at why this is true.

In order for us to prove this, we need to show that we can create an adjacent transposition which means to switch two adjacent tiles. This would then show that we would be able to take any permutation of this puzzle and swap the tiles until we got to the desired configuration such as the identity. Earlier when we were looking at solving the last four tiles of the [4, 20] puzzle, we had seen that we could transpose adjacent tiles. We will be using this idea to prove that this puzzle is solvable. Since we aren't talking about a specific puzzle, the algorithm is going to look slightly different.

We will start off with the identity permutation of a [t, n] puzzle.

$$\cdots \ (1\ 2 \cdots t)\ \ t{+}1\ \ \cdots\ \ n$$

In this example, we want to use an algorithm to transpose tile 1 and tile 2 to show that we can perform an adjacent transposition and have this permutation,

$$\cdots \ (2\ 1 \cdots k)\ \ k{+}1\ \ \cdots\ \ n$$

In this algorithm, we will be using moves called swap-translations. Actually, we have already used swap-translations before in a previous section, but we didn't actually give them a name. A swap-translation is basically a flip followed by a shift of one unit. Of course, the shift has to be given a direction, so we call them counterclockwise and clockwise swap-translations.

Now that we have the basics, we can begin the algorithm. Here are the steps:

1. Move tiles 1 and 2 to the left side of the turnstile, creating the permutation

$$1\quad 2\quad (3\ \ \cdots\ \ t\quad t{+}1\quad t{+}2)\quad t{+}3\quad \cdots\quad t$$

2. Fix the t−1 element to the leftmost position of the turnstile. To make this easier to visualize, we can think about the [4, 20] puzzle. In this case, we would fix the t − 1 element which is tile 3 to the leftmost position of the turnstile.

3. Perform n − t − 1 counterclockwise swap-translations to get to this permutation,

$$(t{+}1\quad t\quad \cdots\quad 1)\quad 2\quad \cdots\quad n$$

4. Perform t/2 + 1 swap-translations and each time you make one, change the direction of the translation starting with a counterclockwise translation. If t ≡ 0(mod4) the

permutation will look like this,

$$(\cdots \quad 2 \quad 1 \quad \cdots) \quad \cdots \quad n$$

and if $t \equiv 2(\text{mod}4)$, then the configuration will look like this,

$$(\cdots \quad 1 \quad 2 \quad \cdots) \quad \cdots \quad n$$

5. Perform $t/2$ swap-translations, again alternating the directions of the translations, starting with a clockwise translation if $t \equiv 0(\text{mod}4)$ and a counterclockwise translation if $t \equiv 2(\text{mod}4)$. This will bring us to this permutation,

$$2 \ (1 \quad \cdots \quad t \quad t+1) \quad t+2 \quad \cdots \quad n$$

6. If we perform one more clockwise translation, we will have this permutation which is where we wanted to end up,

$$\cdots \quad (2 \quad 1 \quad \cdots \quad k) \quad k+1 \quad \cdots \quad n$$

By doing this, we have proven that we can perform adjacent transpositions which means that we can solve any puzzle where n is even and $t \equiv 0(\text{mod}4)$ or $t \equiv 2(\text{mod}4)$. [3]

4.2 Unsolvable Puzzles

My first example of an unsolvable puzzle is a [3,14] puzzle. In this case, there are an even amount of tiles in total, and an odd amount of tiles in the turnstile.

	14	**1**	**2**	**3**	4	
13						5
12						6
	11	10	9	8	7	

	1	**2**	**3**	**4**	5	
14						6
13						7
	12	11	10	9	8	

These are two permutations of this puzzle. Inside the turnstile, you will always have either an EOE or an OEO. When you perform the turnstile flip, this is still the case. Therefore we will never be able to have two evens or two odds next to each other. Thus, since we can't generate all of the permutations of S_n, this puzzle is not solvable.

5 Fewest Number of Moves

Upon doing more research on the fewest number of moves it takes to "solve" a puzzle, I have concluded that it's not as straightforward as we had hoped. We can conclude that there is no way to determine whether a puzzle is sufficiently scrambled. You can have two "scrambled" puzzles where the only difference is that two tiles are switched, but one may be closer to being solved than the other. Also, in many cases it depends on what your goal is for a solved puzzle. You may want to solve the puzzle by having the tiles go in ascending order, descending order, or you may have a completely random permutation that you want your puzzle to be in. In any of these cases, it is not possible to find what the "fewest" number of moves it would take to make this possible.

6 Other Puzzles

There are many other puzzles that relate to the Topspin. Some examples are the Rubik's cube, the Backspin, and the 15-Puzzle.

6.1 Rubik's Cube

We've all tried and failed at solving the Rubik's cube. Some of us may have been lucky enough to memorize the moves needed to solve it, but for a mathematician, that may not be enough. It's a great feeling to be able to know how to solve a puzzle because you know how it works. Now knowing how the Topspin works and that the functionality of the two puzzles are similar, it'll be interesting to go back and try to solve the Rubik's cube with a new view of it.

6.2 The 15-Puzzle

At first glance, the 15-Puzzle may not seem familiar, but if we changed the numbers to be different parts of a picture, then it starts to seem like a puzzle we've probably all played at one point or another. The 15-Puzzle is a 4x4 grid puzzle with 15 tiles numbered 1-15. There is no 16th tile because that hole is left so that the tiles next to the hole can move around. The object of this puzzle is to either put the numbers back in order from left to right and top to bottom, or if the puzzle is of a picture, to restore the picture back to the original formation.

6.3 The Backspin

The Backspin, which seems to be a spin-off of the original Topspin, is a round disk with two sides that rotate with respect to each other. On each side, there are six colored areas that hold colored balls. Each of the twelve areas can hold 3 balls and one of them holds only two to allow for balls to move around. Once the balls are scrambled around, the object of solving the puzzle is to get the colored balls back in the colored areas of each side of the disk.

6.4 Other Topspin Puzzle

Upon more research of other puzzles, there was an online version of a Topspin that had lettered tiles instead of numbered ones. You are able to play it online with different keys corresponding to different moves of the puzzle. After some attempts at using the algorithms for the [4, 20] puzzle, it was concluded that a different algorithm was needed. The reasoning behind this is most likely because there are 26 total tiles instead of 20. Solving the first 22 tiles was the same as the original Topspin, but when trying to solve the last four tiles, it didn't some out to be the same. Further research will be done to try to figure out how to solve the last four tiles of the puzzle [1].

7 Conclusion

After all this, we can see that our first glance at a puzzle is a clueless one. We may not know the true depth that this puzzle empowers or the complexity of it. Puzzles were something that we left for the geniuses to solve because it was too hard or too complex. It doesn't take a mathematician to solve a puzzle, but it may help to have the knowledge that we do to back us up. It'll be interesting to try to solve other sequence puzzles after having mastered this one. Mastering puzzles won't just stop at the Topspin, but will continue to the different puzzles we come across in the future. It's amazing that we can go from having a hands on puzzle to having the same puzzle being used and played with technology. Who knows what type of puzzles we'll have in the future with the rate of change in technology we have going on today. With having all of this math knowledge, we will never look at a puzzle as just a simple toy or a complex phenomena that can only be figured out by math geniuses, but something to be challenged by.

Bibliography

[1] Archer, Mike. ”Top Spin.” Www.MikeArcher.org.uk - Top Spin. N.p., n.d. Web. 14

Dec. 2015.

[2] Bennett, Curtis. ”TopSpin on the Symmetric Group.” The Rose-Hulman Undergraduate Math- ematics Journal 8.2 (2000): 11-15. Web.

[3] Kaufmann, Samuel. ”A Mathematical Analysis of The Generalized Oval Track Puzzle.” Rose- Hulman Undergraduate Mathematics Journal 12.1 (2011): 71-90. Rose-Hulman Institute of Technology. Web.

[4] Mulholland, Jamie. ”Permutation Puzzles: A Mathematical Perspective, 15 Puzzle, Oval Track, Rubiks Cube and Other Mathematical Toys.” Simon Fraser University. (2015). Lecture Notes.

[5] Rowland, Todd and Weisstein, Eric W. ”Group.” From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/Group.html

[6] Wilbur, Eric. ”Topspin: Solvability of Sliding Number Games.” The Rose-Hulman Undergrad- uate Mathematics JournalTopspin: Solvability of Sliding Number Games 2.2 (2011): 1-8. Web