GUARANTEED TO WIN: OPTIMAL STRATEGIES FOR DISCRETE BIDDING GAMES

Honors Thesis

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By

Sophia Brunet

Professor Pedro Poitevin
Faculty Advisor
Mathematics Department

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1 Introduction

Many of us are familiar with two player games, such as Tic-Tac-Toe or chess, where each player alternates taking turns. Players compete against each other, strategically making a move once it's their turn. The goal of the game is simply to "win", depending on the rules of the game. We can add an extra layer to these games that creates some mathematical questions.

Instead of alternating turns, players are now "bidding" to make a move. Not only does this add more competition, strategy, and excitement to the game, but it also adds mathematical intricacies. We call these *Richman games*, studied by David Richman in the 1980s. In Richman games, players make a bid (or auction)[1] of a nonnegative number of chips to make a move. The player that bids the most plays their turn, and then "pays" their chips to the other player.

By studying Richman games, this paper will explore the optimal bidding strategies to maximize game play. The goal of each player is to win the game - not have the most amount of chips. In order to win the game, players need to have bidding strategies to ensure they are making moves. The proportion of chips a player has in their possession at a certain point, or *critical threshold*, is crucial within bidding games. We will explore how to find the critical threshold for games, and how it optimizes a player's chance of winning (also referred to as *winning strategies*). We will also dissect the use of the *tie-breaking advantage* when two players bid the game amount of chips. Through these strategies, we will explore a game of bidding Tug O' War and applications to more extensive games, such as bidding Tic-Tac-Toe.

2 Rules and Regulations

2.1 How to Play

When playing a bidding game, there are a set number of chips at the beginning of the game. Each player starts with a certain amount of chips. A *bid* is the amount of chips a player is willing to give to the opponent to make a move within the game. The player that bids the most chips wins the bid and takes the turn.

The rules of the games are as follows:

1. Players "bid" by writing the number of chips on a piece of paper. The "bids" are revealed simutaneously.

2. The player that bids the most amount of chips makes a move, and forfeits their chips to their opponent.

For example, say two players, Alice and Bob, are playing a game with bidding. There are 20 chips in total. Say Alice and Bob each start with 10 chips. Let Alice bid 4 chips and Bob bid 3 chips. In this scenario, Alice wins the bid. Therefore, she takes her turn, and Alice is left with 6 chips and Bob now has 14 chips.

Similar to the familiar games mentioned above, the goal of bidding games is to win the game. Say the game is bidding Tic-Tac-Toe. The players are aiming to get 3 in a row. Whichever player gets 3 in a row first wins the game - the chip count becomes irrelevant once someone wins. The chips are simply what allows players to make their move - they hold no value once the game has ended.

2.2 Value of a Bid

Richman games involve *real-valued bidding*[2]. However, we will focus on a variation called *discrete bidding*.

Definition 2.1. *A Richman game using **discrete bidding** allows players to only bid an amount of chips that are natural numbers, including 0.*

As oppossed to real-valued bidding games, players may only bid natural numbers. If each chip has a value of 1, it is not possible to bid $\frac{1}{2}$ of a chip or π chips. It is also easier to keep track of the amount of chips each player has when using natural numbers. Players cannot bid a negative number, as it is impossible to bid less than zero chips. Lastly, a player's bid cannot be greater than the amount of resources they have; Alice may not bid 9 chips if she has 6. It is also conveinent to keep the number of chips in the game relatively low. This extra layer of bidding to traditional two-player games not only adds fun but allows us to explore the mathematics of optimal strategies.

2.3 Tie-Breaking Advantage

Since we are working with a discrete value of chips, it is possible for two players to bid the same amount, resulting in a tie. One player is designated to start with the *tie-breaking advantage*. For simplicity in the game, the player starting with the tie-breaking advantage is determined by a coin flip at the beginning of the game. In terms of value, the tie-breaking advantage is positive, but less than the value of a bidding chip. So in this case, the value of the tie-breaking advantage is between 0 and 1.

The player with the *tie-breaking advantage* has two options when the bids are tied:

1. Declare themselves the winner of the bid. Then, they give the *tie-breaking advantage* and the bidded chips to the other player, and make the move.

2. Declare the other player the winner. Hence, keeping the tie-breaking advantage. In this case, the other player makes the move and gives their bidded chips to the player with the advantage.

The tie-breaking advantage is represented by $*$. For example, say that Alice and Bob each start with 10 chips, and Alice has the tie-breaking advantage. Both players bid 4 chips.

Alice now has two options:

1. Alice uses the tie-breaking advantage to make the move. She then gives her chips to Bob and the tie-breaking advantage, but she gets to make a move on the board. The new chip count is Alice: 6, Bob: 14*.

2. Alice keeps the tie-breaking, allowing Bob to make the move. Alice receives Bob's chips and keeps the tie-breaking advantage. The new chip count is Alice: 14*, Bob: 6.

Note that we will talk about optimal strategies of bidding games for Alice's perspective of playing.

The tie-breaking advantage is crucial when it comes to optimal strategies in bidding games. Players must decide when to use the tie-breaking advantage or to keep it, in order to optimize their chance of winning the bid, and then the game. The tie-breaking advantage is never disadvantageous to a player.

Lemma 2.1. *If Alice wins $G(a, b^*)$, then she also wins $G(a^*, b)$.*

[2]

Proof. $G(a, b^*)$ is the game in which Alice has a chips and Bob has b chips and the tie-breaking advantage (denoted by *). During $G(a^*, b)$, Alice will play the game as if she does not have the tie-breaking advantage, as she will not change any bidding strategies. The first time that the bids are tied, Alice will then use the tie-breaking advantage. This brings us to the game $G(a, b^*)$, which was the game that she won in the premise. □

We already knew that Alice won the game when she had the tie-breaking advantage. Once Alice had the tie-breaking advantage and used it, we had the original game that she won. This lemma demonstrates that the tie-breaking advantage never hurts a player's chance of winning and always has a positive value. Although Bob ended the game with the tie-breaking advantage, his strategy was not affected, as Alice already won the original game of $G(a, b^*)$.

In short, if Alice wins the game without the tie-breaking advantage, she will then win the game with the tie-breaking advantage.

As mentioned above, players want to have the tie-breaking advantage, as the value is always positive and it will never hurt their optimal moves. However, it has less value than another bidding chip.

Lemma 2.2. *If Alice wins $G(a^*, b+1)$, then she also wins $G(a+1, b^*)$.*

[2]

Proof. Let's look at $G(a+1, b^*)$. Alice's strategy for the game is to play as if she has a chips and the tie-breaking advantage. However, instead of bidding k amount of chips, Alice will bid $k+1$ chips. So, instead of Alice and Bob bidding $k - to - k$ chips and Alice having to use the tie-breaking, Alice bids $k+1$ chips. In this case, Alice bid one extra chip, the bids are not tied, and she wins the bid. This brings the bids to $k+1 - to - k$, allowing Alice to win the bid and make the move. □

In the game $G(a^*, b+1)$, the results of the bids are the same if Alice bids $k+1$ chips or k chips with the tie-breaker. The extra chip that Alice bid had more value than the tie-breaking advantage since Bob has $k+1$ chips, and therefore she has a winning strategy.

The tie-breaking advantage adds value to the total number of resources that a player has, but not as much value as another chip. In later sections, we will see how the tie-breaking advantage can be applied to gameplay.

3 Critical Threshold

Critical threshold is an important component of bidding games, as it determines who has a winning strategy. For the purpose of this paper, we will look at the critical threshold and winning strategies from Alice's perspective (we want Alice to win the game).

Definition 3.1. *The **critical threshold** is the proportion of chips that a player needs to have a winning strategy.*

- *If the proportion of chips that the player has is greater than the critical threshold, the player has a winning strategy.*
- *If the proportion of chips that the player has is less than the critical threshold, the player does not have a winning strategy (and their opponent does).[2]*

We denote the critical threshold of a game as $R(G)$. Since the critical threshold is a proportion, it will always be between 0 and 1. Oftentimes, we want to know the critical threshold at a specific point in a game. Let v be a position in a game. For instance, v could be the third move in a game of Tic-Tac-Toe. Each position has a specific critical threshold, which we denote as $R(G_v)$. When we know the critical threshold at each position, a player that has more chips than the critical threshold can determine how many chips to bid, and therefore has a *bidding strategy.* Players do not want to bid a number of chips that would cause them to go below the critical threshold.

According to *Combinatorial Games under Auction Play* [3] , the critical threshold of a specific position, or *vertex* v is an average of the maximum and minimum values of the position. For the purpose of this paper, we will always know the maximum and minimum values. So,

$$R(G_v) = \frac{R_{min}(G_v) + R_{max}(G_v)}{2}.$$

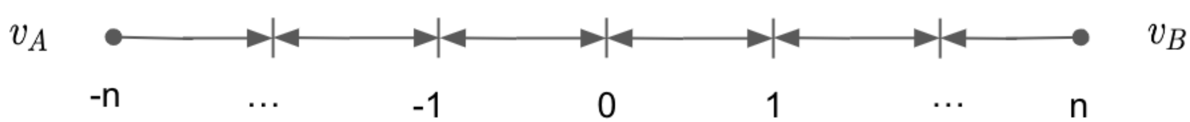
This formula allows us to determine the critical threshold at each point in the game. We often work backwards, starting with the endpoints of a game, to find the critical thresholds. When we know the critical threshold at each point, players will know exactly what proportion of chips they need at each position to have a winning strategy. This will allow players to be strategic with how many chips they bid.

In later sections of this paper, we will explore the critical thresholds of specific games.

3.1 Calculating the Critical Threshold

As seen above, the critical threshold of a specific point of a game is the average of the maximum and minimum values of the vertex. We often start with the end positions of the game and work backward from a directed graph to calculate the critical threshold at each point. An example using bidding Tic-Tac-Toe will be discussed in future sections.

A *directed graph* is a graph in which the edges, or points, have directions. Directed graphs are important in illustrating bidding games, as it represents the different moves that can be played. For example, here is a directed graph of a bidding game, *Tug O' War* that will be explored later in the paper. We can see the arrows directing us to what positions a player can move to.



Essentially, there is a winning vertex for Alice, v_A, and a winning vertex for Bob, v_B. Alice's goal is to end at v_A, while Bob's goal is to end at v_B. The moves throughout the game change the position on the graph. When the position of the game reaches v_A, the critical threshold is 0, $R(v_A) = 0$ as Alice has won and does not need any more resources to win the game. Then, the critical threshold at v_B is 1, $R(v_B) = 1$, as Alice would need all of the resources to win the game (which is impossible at this point, since Bob has already won). A directed graph shows us the different positions players can take in the game.

4 Let's Play!

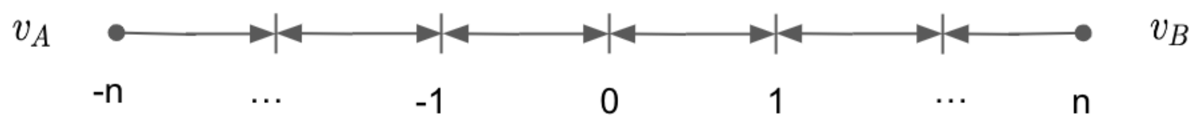
In this section, we look at the application of the tie-breaking advantage, and more importantly, critical threshold in bidding games. We will first look at a simple game of bidding

Tug O' War, and then bidding Tic-Tac-Toe.

4.1 Bidding Tug O' War

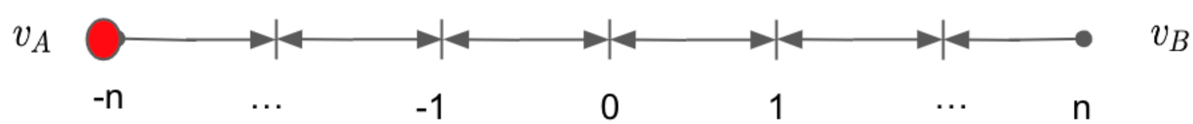
This game of Tug O' War is different than the where players stand on each end of a rope and aim to pull it to a certain distance.

In *Bidding Tug O' War*, there is a graph of length of $2n$, with vertices labeled $-n, ..., -2, -1, 0, 1, 2, ..., n$ from left to right.

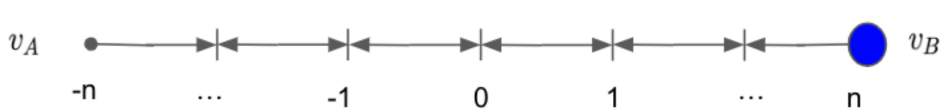


We denote the game as Tug^n, with n being the amount of positions (or vertices) until a winning position starting from 0. Each player has a winning vertex at $-n$ or n. Alice wins at the left-most vertex, -n. Bob wins at the right-most vertex, n.

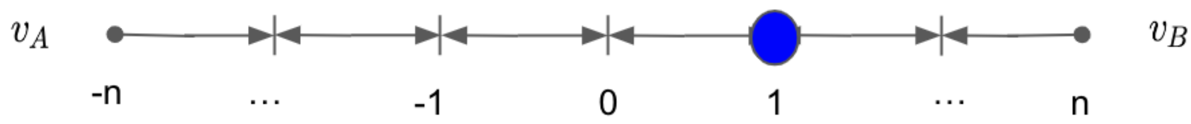
Alice win:



Bob win:



The game starts at 0. Each time a player wins a bid, the position of the game moves on point, or vertex, closer to their winning position. Say Bob wins the first bid. The position of the game is at 1.



The nature of Tug O' War provides us with a simpler game to analyze these mathematical concepts, as players can only move left or right. We could use the original formula for critical threshold. However, there is a simpler way to find the critical threshold at each position for Tug O' War.

Theorem 4.1. *Let k be the position of the game. Suppose G is a directed linear graph with two endpoints, and an edge (v, u) whenever (u, v) is an edge (that is, G is really an undirected graph). Suppose that k is a vertex on some minimal length path from v_A to v_B(from the vertex in which Alice wins to the vertex in which Bob wins). Then $R(k) = \frac{dist(v_A,k)}{dist(v_A,v_B)}$ where $dist(u, v)$ is the length of the shortest path from u to v.*

Let us check that the formula is coherent with formula 3, and that therefore, the critical threshold of each vertex is exactly as it should be in the specific situation of Tug O' War.

Remark:

First, we will calculate $R(G_v) = \frac{R_{min}(G_v)+R_{max}(G_v)}{2}$.

To calculate $R_{min}(G_v)$ and $R_{max}(G_v)$, we use the definition of Theorem 4.1.

So, $R_{min}(G_v) = R(G_{v-1}) = \frac{dist(v-1,v_A)}{2n} = \frac{(v-1)-v_A}{2n}$.

Then, $R_{max}(G_v) = R(G_{v+1}) = \frac{dist(v+1,v_A)}{2n} = \frac{(v+1)-v_A}{2n}$.

Note that $\frac{R_{min}(G_v)+R_{max}(G_v)}{2} = \frac{\frac{(v-1)-v_A}{2n}+\frac{(v+1)-v_A}{2n}}{2} = \frac{\frac{2v-2v_A}{2n}}{2} = \frac{v-v_A}{2n} = R(G_v)$

This theorem demonstrates that the critical threshold for Tug^n is:

$$\frac{\text{distance between the position of the game from the winning position}}{\text{distance from each end point}}.$$

The distance between v_A and position k is $k-(-n)$, or $k+n$. The distance between v_A and v_B is the distance between n and $-n$, or $n-(-n)$ which is equal to $2n$.

Our formal for the critical threshold of a game of Tug^n is

$$R(G_v) = \frac{k+n}{2n}.$$

Let's look at a game with $n = 3$. As a reminder, we are looking at the perspective of Alice, so we will use v_A as our winning position.

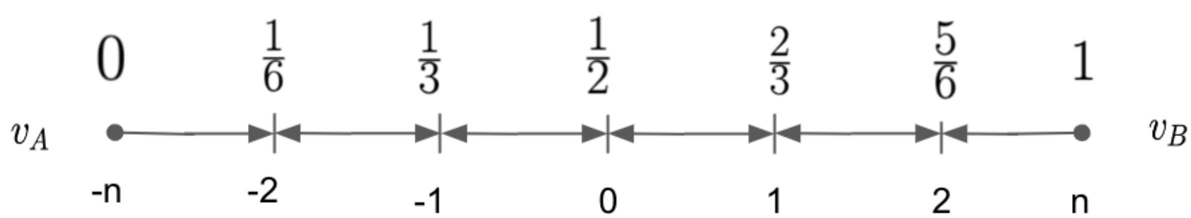
The critical threshold of the Alice's winning position vertex $-n$ is 0, as $R(-n) = \frac{0}{2n} = 0$. In other words, Alice wins at position $-n$, and therefore does not need any more chips to have a winning strategy since she has already won.

For Alice, the critical threshold at Bob's winning position vertex n is 1, as $R(n) = \frac{2n}{2n} = 1$. Alice would need all of the chips to have a winning strategy when the position of the game is at n (a win for Bob), which would not be possible as Bob has already won the game.

Using the critical threshold formula above, we can find the critical threshold at each point for Tug^3. Say we are at position $k = -2$. The critical threshold is as follows:

$\frac{-2+3}{6} = \frac{1}{6}$.

So, at $k = -2$, Alice needs $\frac{1}{6}$ of the total number of chips to have a winning strategy. The critical thresholds at each point is labeled below, found by our formula $R(G_v) = \frac{k+n}{2n}$:



It's no surprise that as Alice gets closer to her winning position, v_A, she needs fewer chips, and therefore a lower critical threshold. This holds true as Alice needs more chips as Bob gets closer to winning.

Let's look at a game with a specific chip count.

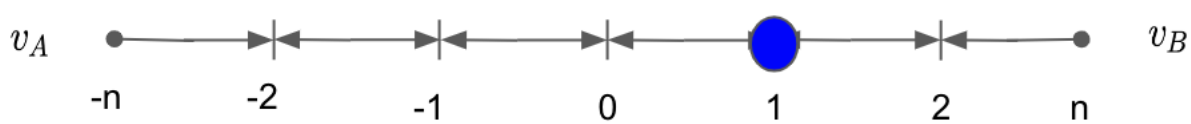
Proposition 4.1. *Suppose Bob's total number of chips is less than n. Then Alice wins Tug^n if and only if her total number of chips is at least $(n-1)^*$.*

[2]

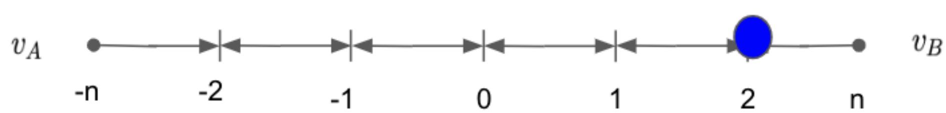
Since $n = 3$, Alice starts with 2 chips and the tie-breaking advantage and Bob starts with 2 chips. We denote this game $G(2^*, 2)$.

According to *Discrete Bidding Games* [2] , Alice has a winning strategy for this game if she bids 0 every time she has the tie-breaking advantage (and uses it in the case of a tie), or bids 1 when she does not have the tie-breaking advantage. A move by Alice is represented by a red tile on the position, and a move by Bob is represented by a blue tile.

Move 1: Alice has the tie-breaking advantage, and therefore bids 0 chips. Bob bids 1 chip. Bob wins the bid, moving the position one vertex closer to his winning position n. The position of the game is $k = 1$. Alice has 3 chips and the tie-breaking advantage and Bob has 1 chip, denoted $G(3^*, 1)$.



Move 2: Alice bids 0 chips and Bob bids 1 chip. Bob wins the bid, moving the position one vertex closer to n. The position of the game is now $k = 2$. Alice has 4 chips and the tie-breaking advantage, Bob has 0 chips, $G(4^*, 0)$.



Move 3: Alice still has the tie-breaking advantage and therefore bids 0 chips. Bob has no more chips and is forced to bid 0. The bids are tied; Alice uses the tie-breaking advantage to win this bid. The position is now one vertex closer to $-n$, Alice's winning position. The new position is $k = 1$. Alice still has 4 chips, but Bob now has 0 chips plus the tie-breaking advantage, $G(4, 0^*)$.

As the game continues, the *weight* of Alice's position is getting closer to $-2n$. We define the weight as the number of the current position minus the number of chips Alice has (including the tie-breaking advantage) [2] . The weight of the position determines the chances a player will win. As the weight of Alice's position gets closer to $-2n$, the higher the chance she has of winning the game. This game demonstrates the use of Lemma 2.2, as the tie-breaking advantage affects the weight of the position:

Move 1: The position is $k = 1$, and Alice has 3^* chips. The weight is -2^*.

Move 2: The weight of Alice's position at $k = 2$ is -2^*. The weight did not change.

Move 3: The weight of Alice's position at $k + 1$ is -3.

With this strategy, the weight of Alice's position continues to get closer to $-2n$, and therefore Bob can never win.

This strategy works for the particular game described in Proposition 4.1. We can determine the chips needed for Alice to have a winning strategy for all chip counts.

4.2 Bidding Tug O' War with All Chip Counts

In order to show that Alice has a winning strategy for Tug^n for all chip counts, we will reference Theorem 3.13 from Develin and Payne. :

Theorem 3.13. If Alice has a stable winning strategy for $G(a^*, b)$ then she also has a stable winning strategy for $G(a+m^*, b+\bar{m}$. Similarly, if Alice has a stable winning strategy for $G(a, b^*)$ then she also has a stable winning strategy for $G(a+m, b+\bar{m}^*)$. [2]

Note: Alice has a *stable winning strategy* if the critical threshold at the position k is as small as possible [2]. In other words, we want Alice to have the least amount of chips possible to still reach the critical threshold, as it will maximize her chance of winning.

In the case of our game Tug^n, $m = \bar{m} = n$. We know this from Section 3.6 in Develin and Payne's paper [2].

Corollary 4.2. *Let a, b, k, and k' be nonnegative integers, with a and b less than n. Then Alice wins Tug$^n(kn + a, (k'n + b)^*)$ if and only if k is greater than k'. Furthermore, Alice wins Tug$^n(kn + a^*, k'n + b)$ if and only if either k is greater than k' or k is equal to k' and a is equal to $n - 1$.*

[2]

Proof. Let $k = k'$. If Alice wins $G(a^*, b)$, Alice also wins $G(a+m^*, b+\bar{m})$. Since $m = \bar{m} = n$, Alice and Bob both gained the same amount of chips, and therefore have the same winning strategies. Hence, Alice also wins $G(a + n^*, b + n)$. Applying Theorem 3.13, Alice also wins $G(a+n+n^*, b+n+n)$, which is the same game as $G(a+2n^*, b+2n)$. We can keep adding n amount of chips to Bob and Alice, until the game is $G(a + kn^*, b + kn)$. Applying Theorem 3.13 again, we see that Alice will win $G(a + (k-1)n^*, b + (k-1)n)$. Inductively, we can see that when $k = k'$ and Alice wins $G(a^*, b)$, she inductively wins Tug$^n(kn + a^*, k'n + b)$.

Let $k > k'$. In Tug$^n(kn + a, k'n + b^*)$ Bob now has the tie-breaking advantage. However, since $k > k'$, Alice has a greater multiple of chips. Alice then has more resources by Lemma 2.2. So, Alice wins any game $G(A, k'n + b^*)$ as long as $A > k'n + b + 1$. □

We used induction to prove that for Tug$^n(kn + a, (k'n + b)^*)$, if $k > k'$, Alice will always have a greater multiple of k chips. For Tug$^n(kn+a^*, k'n+b)$, we know that k can be equal to k'. So in the previous argument, it is not guaranteed that Alice will have a greater multiple of chips. Since $a = n - 1$, and a and b are less than n, the maximum number of chips both Alice and Bob can have are $n - 1$ there are two cases:

- Case 1, $a = b$: If both Alice and Bob have the same number of chips, Alice wins by having the tie-breaking advantage.
- Case 2: $a > b$: Alice may have more chips than Bob. In this case, she has more resources, plus the tie-breaking advantage, and has a winning strategy.

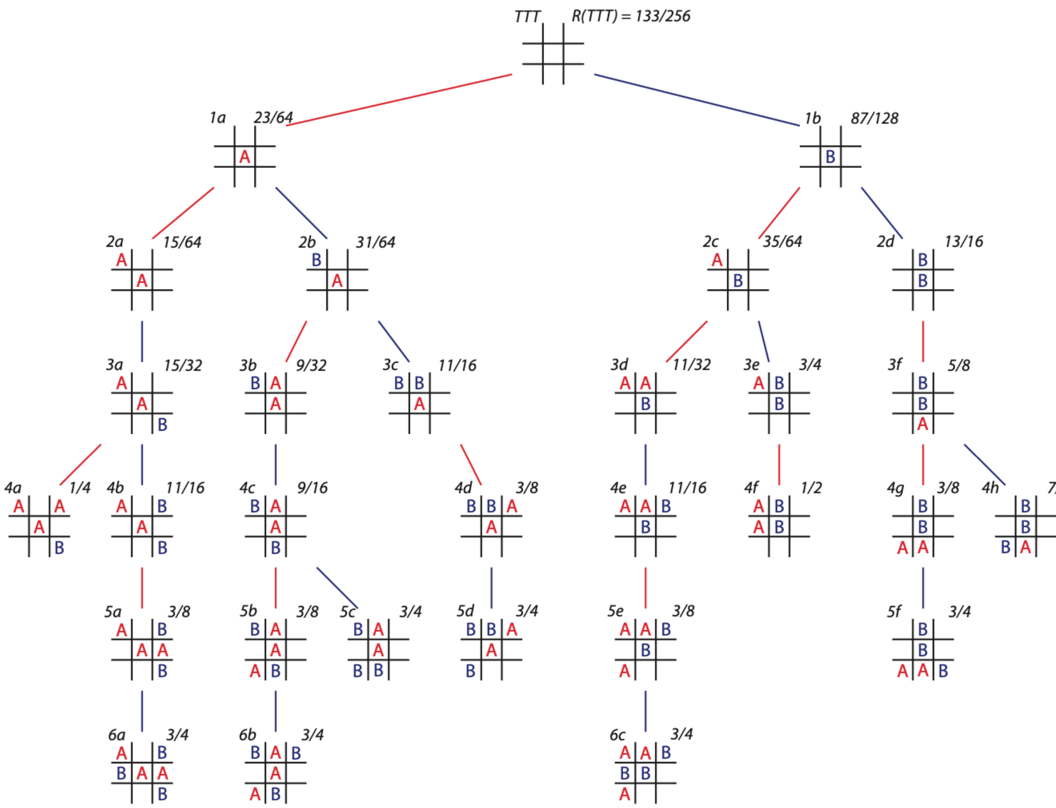
Let's look at an example. We know that a is the amount of chips Alice has, b is the amount of chips Bob has, n is the number of vertices on one half of the directed graph of Tug^n, k is a position on the board, and k' is another position. For our example, let $a = 4, b = 3, k = 7, k' = 6$, and $n = 3$. This satisfies the conditions of Corollary 4.2. So, we have the game $\text{Tug}^n(31, 21^*)$, which Alice wins. She also wins $\text{Tug}^n(31^*, 21)$ since the tie-breaking adds more value. In this case, she would place as normal until Alice and Bob tie their bids. Alice would use the bid, and it brings us back to the original game.

Corollary 4.2 aims to demonstrate how Alice has a winning strategy with all chip counts.

Bidding Tug O' War is a simple game used to discuss bidding and winning strategies, Players are only moving left or right, depending on their winning vertex. There are other bidding games that are more complex and use bidding strategies.

4.3 Bidding Tic-Tac-Toe

Bidding Tic-Tac-Toe, or TTT, adds another layer to our discrete bidding games. Instead of just moving left or right, there is now strategy to players' moves to try to get 3 X's or O's (or in our case, A's for Alice and B's for Bob) in a row.



[2]

We can see that the diagram representing the gameplay of TTT is more complex than Tug^n, as there are more options for making a move. This diagram does not represent all possible positions of gameplay, as there are many ways for players to make their move. The fractions at each position are the critical thresholds for each position. Although the game may be more complex, our bidding strategies still optimize gameplay.

We can see that our formula for critical threshold holds true for TTT, working backwards from end positions. The critical threshold at each position is labeled on our diagram, but let's see how the formula holds true. Let's see how the critical threshold at position *4c* is $\frac{9}{16}$.

Using formula 3, let $R_{min}(G_v) = \frac{3}{8}$ and $R_{max}(G_v) = \frac{3}{4}$, as seen from positions *5b* and *5c* (reminder, we are working backwards). So,

$$R(4c) = \frac{\frac{3}{8}+\frac{3}{4}}{2} = \frac{9}{16}.$$

This is just one example as to how critical threshold works in more complex games that have various options to move. More information can be found in or could be used in further research.

5 Conclusion

The differences in Bidding Tug O' War and Bidding Tic-Tac-Toe are not in the mathematical concepts and theorems but in the complexities of the games themselves. Tug^n is a simpler game than TTT since there is not a strategy to the actual gameplay, only bidding strategies. TTT involves strategy in placing and X or O on the board, trying to block your opponent from winning, etc. The same concepts within this paper can be applied to other games, such as chess, checkers, or Ultimatum, with the additonal layer of bidding. We could also explore how bidding strategies are affected when we use real-valued bidding instead of discrete bidding. More information and research on these games can be found in the original paper.

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References

[1] G. Avni, T. Henzinger, *An Updated Survey of Bidding Games on Graphs*, 47th International Symposium on Mathematical Foundations of Computer Science, (2022).

[2] M. Develin, S. Payne, *Discrete Bidding Games*, (2008). arXive:0801.0579v1 [math.CO].

[3] A. Lazarus, D. Loeb, J. Propp, W. Stromquist, D. Ullman, *Combinatorial Games under Auction Play* (1997).